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The Stable $\mathbb{A}^1$-Connectivity Theorems

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1. Introduction

Let S be a separated noetherian scheme of finite Krull dimension, and let Sm_S be the category of smooth separated, finite type S-schemes. We denote by Sp the category of S^1-spectra [6] and by $Sp^{S^1}(Sm_S)$ the category of sheaves of S^1-spectra on Sm_S in the Nisnevich topology [31,32]. Such a sheaf of spectra E is said to be (-1)-connected if, for each integer $n<0$, its n-th homotopy sheaf $\pi_n(E)$ vanishes. Finally, we denote by $L_{\mathbb{A}^1}: Sp^{S^1}(Sm_S) \to Sp^{S^1}(Sm_S)$ the (stable version of the) $\mathbb{A}^1$-localization functor of [31].

DEFINITION 1. Given S as above, we say that the stable $\mathbb{A}^1$-connectivity property holds over S if the $\mathbb{A}^1$-localization functor $L_{\mathbb{A}^1}$ preserves (-1)-connected S^1-spectra.

Our purpose is to address the:

CONJECTURE 2. *The stable $\mathbb{A}^1$-connectivity property holds over any regular base scheme S.*

Our main result is that the stable $\mathbb{A}^1$-connectivity propertyholds when the base scheme S is the spectrum of a field. We will discuss consequences of that property and also provide some general tools which might hopefully help to prove more general cases[1] of the stable $\mathbb{A}^1$-connectivity property, e.g. when S is essentially smooth over some field. In the sequel, k will always denote a fixed field. We will thus prove:

THEOREM 3 (cf. 6.1.8). *Assume $S = Spec(k)$. The $\mathbb{A}^1$-localization of a (-1)-connected S^1-spectrum is still (-1)-connected.*

[1] J. Ayoup has disproved the previous conjecture in case the dimension of S is ≥ 2; the dimension 1 case is still open.

The $\mathbb{A}^1$-localization functor is described in Section 4 and the proof of the Theorem 3 appears in Section 6. In Section 4 we develop some general properties of base change. Sections 1 and 2 are recollections on well known facts.

In the case k is a perfect field, the theorem can be proven (see [27]) using the homotopy purity theorem of [31]. In the remaining case where k is not perfect, thus is infinite, we must use Gabber's presentation lemma [13, Lemma 3.1] and [9].

The following notion will play an important role in this paper:

DEFINITION 4. A sheaf of abelian groups M in the Nisnevich topology on Sm_S is called strictly $\mathbb{A}^1$-invariant if for any $X \in Sm_S$, the canonical morphism

$$H^*_{Nis}(X; M) \to H^*_{Nis}(X \times \mathbb{A}^1; M)$$

is an isomorphism.

We make the following observation which justifies the introduction of the previous notion:

LEMMA 5 (cf. 6.2.9). *Assume that the stable $\mathbb{A}^1$-connectivity property holds over S. Then:*

(1) *For any sheaf E of S^1-spectra over S and any integer $n \in \mathbb{Z}$, the sheaves*

$$\pi_n^{\mathbb{A}^1}(E) := \pi_n(L_{\mathbb{A}^1}(E))$$

are strictly $\mathbb{A}^1$-invariant.

(2) *Let E: $(Sm_S)^{op} \to Sp$, $U \mapsto E(U)$ be a presheaf of S^1-spectra over S which has the B.G. property for distinguished squares (see 3.1.6 and 3.1.8, or [31]) and the $\mathbb{A}^1$-invariance property: for any $U \in Sm_S$, the morphism $E(U) \to E(U \times \mathbb{A}^1)$ is a stable weak equivalence. Then for any $n \in \mathbb{Z}$, the Nisnevich sheaf $\mathcal{H}^n(E) = \pi_{-n}(E)$ associated to the presheaf $U \mapsto E^n(U) := \pi_{-n}(E(U))$ is a strictly $\mathbb{A}^1$-invariant sheaf.*

The well-known examples of strictly $\mathbb{A}^1$-invariant sheaves are constant sheaves, sheaves represented by semi-abelian S-schemes over a general base scheme. Over k the sheaves of unramified Milnor K-theory [19,35] and more generally the sheaves of the form $X \mapsto A^0(X; M)$ for each Rost's *cycle module* M, see [35]. When k is perfect, Voevodsky's *homotopy invariant sheaves with transfers* [40] are strictly $\mathbb{A}^1$-invariant. These two types of examples in fact agree by [10,11].

For instance, for any $n \in \mathbb{N}$, the sheaf $\mathcal{K}_n$ on Sm_k in the Nisnevich topology associated to the presheaf $X \mapsto K_n(X)$ of Quillen's algebraic K-groups [33] is strictly $\mathbb{A}^1$-invariant. Indeed, following [31,41] the presheaves

$$X \mapsto K_n(X)$$

of Quillen's K-groups can be represented by an S^1-spectrum, a T-spectrum indeed.[2] But this fact is well-known, for instance it comes from a Rost's cycle module [35] as well.

The following example is however new:

COROLLARY 6. *Assume* $char(k) \neq 2$. *Then the sheaf* $\underline{\mathbf{W}}$ *on* Sm_k *in the Nisnevich topology associated to the presheaf* $X \mapsto W(X)$ *of Witt groups [2,20] is strictly* $\mathbb{A}^1$*-invariant.*

We observe that by [3,4] the sheaves associated to the presheaves $X \mapsto W^i(X)$ of Balmer Witt groups vanish for $i \neq 0$ [4] and are equal to $\underline{\mathbf{W}}$ for $i = 0$ [4].

I. Panin told the author he also has a proof of the corollary which elaborates on Voevodsky's technic from [39], as $X \mapsto W(X)$ has some type of transfers, though not being a presheaf with transfers in the sense of *loc. cit.*. Corollary 6 is used in the computations in [28,29].

Proof. By [16] there is a sheaf of S^1-spectra $X \mapsto KW(X)$ on Sm_k which represents[3] Balmer's Witt groups. □

Remark 7. By the lemma above analogous results hold over a regular base scheme S on which the stable $\mathbb{A}^1$-connectivity propertyholds; indeed, Quillen K-groups are still represented by an S^1-spectrum. If moreover 2 is invertible in S, the same holds for the sheaf $\underline{\mathbf{W}}$ associated to Witt groups in the Nisnevich topology on Sm_S by [16]; clearly an analogous result holds also for the sheaves associated to hermitian K-theory.

A general consequence of the stable $\mathbb{A}^1$-connectivity propertyover S is that the strictly $\mathbb{A}^1$-invariant sheaves form an abelian category $\mathbb{A}b_{\mathbb{A}^1}(S)$, in fact an abelian sub-category of the category $\mathbb{A}b(S)$ of sheaves of abelian groups on Sm_S for which the functor $\mathbb{A}b_{\mathbb{A}^1}(S) \subset \mathbb{A}b(S)$ is exact, see Lemma 6.2.13 below. In a very precise sense, these objects are the analogues in the *motivic algebraic topology* for smooth S-schemes of the "discrete" abelian groups in classical algebraic topology. Here is what we mean.

First observe that if Sm_S denotes the category of differentiable manifolds endowed with the classical topology, the category of strictly $\mathbb{R}$-invariant

[2] See also [16].
[3] At least for X affine.

sheaves is just equivalent to the category of abelian groups. A classical[4] S^1-spectrum E admits a Postnikov tower

$$\{P^n(E)\}_{n\in\mathbb{Z}} = \left\{\cdots \to P^n(E) \to \cdots \to P^{-1}(E) \to \cdots\right\}$$

for which the fiber of $P^n(E) \to P^{n-1}(E)$ is the Eilenberg-MacLane spectrum $H(\pi_n(E))$ associated to the n-th stable homotopy group $\pi_n(E)$ of E.

In the world of $\mathbb{A}^1$-homotopy theory over S [25,31,41], we deduce, when the stable $\mathbb{A}^1$-connectivity propertyholds over S, that any sheaf of S^1-spectra E on Sm_S (in the Nisnevich topology) admits a canonical Postnikov tower as above in which the fiber of $P^n(E) \to P^{n-1}(E)$ is the Eilenberg–MacLane spectrum $H(\pi_n^{\mathbb{A}^1}(E))$ associated to the n-th stable $\mathbb{A}^1$-homotopy sheaf $\pi_n^{\mathbb{A}^1}(E) = \pi_n(L_{A^1}(E))$ of E, which is a strictly $\mathbb{A}^1$-invariant sheaf. In other words, there is a t-structure [5] on the stable $\mathbb{A}^1$-homotopy category of S^1-spectra whose heart is $\mathbb{A}b_{\mathbb{A}^1}(S)$.

That (potential) t-structure is called the *homotopy t-structure*. It is the analogue in the stable $\mathbb{A}^1$-homotopy theory of S^1-spectra of Voevodsky's homotopy t-structure for the triangulated category $DM^{\mathrm{eff}}(k)$ over a perfect field k [40].

Remark 8. An obvious abelian variant of our Conjecture 2 and Theorem 3 is obtained by working with chain complexes of sheaves of abelian groups on Sm_S instead of sheaves of S^1-spectra. Recall that $\mathbb{A}b(S)$ denotes the abelian category of sheaves of abelian groups on $(Sm_S)_{Nis}$. Let us denote by $D(\mathbb{A}b(S))$ its derived category. One defines the notion of $\mathbb{A}^1$-*local chain complex* C_* and $\mathbb{A}^1$-*quasi isomorphisms* in the same way as the notions of $\mathbb{A}^1$-local S^1-spectra and stable $\mathbb{A}^1$-weak equivalences, see Definition 4.1.1. The localization of $D(\mathbb{A}b(S))$ by the class of $\mathbb{A}^1$-quasi isomorphisms is denoted by $D_{\mathbb{A}^1}(\mathbb{A}b(S))$ and will be called the $\mathbb{A}^1$-derived category of $\mathbb{A}b(S)$. One can constructs in the same way the $\mathbb{A}^1$-localization functor

$$L_{\mathbb{A}^1} : D(\mathbb{A}b(S)) \to D(\mathbb{A}b(S))$$

which as usual identifies $D_{\mathbb{A}^1}(\mathbb{A}b(S))$ to the full subcategory consisting of $\mathbb{A}^1$-local complexes.

We then conjecture that this functor preserves non-negative chain complexes over any base S. In fact one can prove that this is implied by Conjecture 2. Our proof of Theorem 3 can be adapted to get the abelian version: the $\mathbb{A}^1$-localization functor over a base field

$$L_{\mathbb{A}^1} : D(\mathbb{A}b(k)) \to D(\mathbb{A}b(k))$$

[4]cf [1,6].

does preserve non-negative chain complexes. We thus get an homotopy t-structure on $D_{\mathbb{A}^1}(\mathbb{A}b(k))$ in that case.

All this is compatible with the case of S^1-spectra through the derived functor of abelianization[5]

$$\mathcal{SH}^{S^1}(k) \to D(\mathbb{A}b(k))$$

Remark 9. An other variant in the spirit of the preceding remark is obtained as follows. Let us denote by $\mathbb{A}b^{\mathrm{tr}}(S)$ the category of sheaves with transfers in the Nisnevich topology over Sm_S in the sense of Voevodsky [40]. In the case of a general base S one using the group of finite correspondences defined in [38]. One proves as in [40] that $\mathbb{A}b^{tr}(S)$ is an abelian category. Let us denote by $DM^{\mathrm{eff}}(S) \subset D(\mathbb{A}b^{\mathrm{tr}}(S))$ the full subcategory consisting of $\mathbb{A}^1$-local chain complexes and by

$$L_{\mathbb{A}^1} : D(\mathbb{A}b^{\mathrm{tr}}(S)) \to DM^{\mathrm{eff}}(S)$$

the left adjoint to this inclusion.

Then the same technics from our proof of Theorem 3 proves the $\mathbb{A}^1$-connectivity property holds for this functor when $S = Spec(k))$. When k is perfect, Voevodsky showed [40] that the functor $L_{\mathbb{A}^1}$ is equal to the Suslin-Voevodsky functor $\underline{C}_*$. This proves the $\mathbb{A}^1$-connectivity property in that case. In case k is no longer perfect, the $\mathbb{A}^1$-localization functor is more mysterious and the $\mathbb{A}^1$-connectivity property in that case is new. We hope, as in conjecture 2, that this property always holds. Of course that picture fits with the one of the preceding remark through the derived functor of "adding transfers":

$$D(\mathbb{A}b(S)) \to D(\mathbb{A}b^{\mathrm{tr}}(S))$$

In case of a base field, the heart of the associated homotopy t-structure is the category of strictly $\mathbb{A}^1$-invariant sheaves with transfers. For k perfect, this is exactly Voevodsky's abelian category of $\mathbb{A}^1$-invariant sheaves with transfers by the results of [39].

One can also deduce from our theorem that the functor

$$DM^{\mathrm{eff}}_{gm}(k) \to DM^{\mathrm{eff}}(k)$$

is a full embedding. This is done in the same way as Voevodsky's proof in the case k is perfect [40]. We don't know whether Voevodsky's cancellation theorem [42] holds in that case.

On the way we will prove a refined form of Theorem 2:

[5] Which is induced by mapping a sheaf of pointed sets to the free sheaf of abelian groups with the relation base point $=0$.

THEOREM 10 (cf. 6.4.1). *Assume the stable $\mathbb{A}^1$-connectivity propertyholds over S. Let X be a smooth S-scheme and $U \subset X$ be an open subscheme such that the complementary closed immersion $Z \to X$ is everywhere of codimension $\geqslant d$ and equidimensional over S (in the sense of [38]). Let X/U denote the obvious quotient sheaf of pointed sets in the Nisnevich topology on Sm_S and let (X/U) denote its suspension S^1-spectrum. Then the $\mathbb{A}^1$-localization $L_{\mathbb{A}^1}(X/U)$ of the S^1-spectrum (X/U) is a $(d-1)$-connected sheaf of S^1-spectra on Sm_S.*

Observe that the case $U = \emptyset$ and $d = 0$ is just Theorem 3. If Z is assumed to be smooth over S we know from the homotopy purity of [31] that X/U is $\mathbb{A}^1$-weakly equivalent to $Th(\nu_i)$, the Thom space of the normal bundle of the closed immersion $i : Z \to X$; this easily implies the Theorem in that case.

Assuming the stable $\mathbb{A}^1$-connectivity propertyholds over S, an easy reformulation of the previous theorem (see Corollary 6.4.6) is the following. For any smooth S-scheme X and any open subscheme $U \subset X$ such that the complementary closed immersion $Z \to X$ is everywhere of codimension $\geqslant d$ and such that $Z \to S$ is a universally equidimensional morphism, and for any strictly $\mathbb{A}^1$-invariant sheaf M on Sm_S the morphism

$$H^n_{Nis}(X; M) \to H^n_{Nis}(U; M)$$

is an isomorphism for $n \leqslant d-2$ and a monomorphism for $n = d-1$.

This property is wrong for a general M if one removes the hypothesis that Z is equidimensional. For instance, given a strictly $\mathbb{A}^1$-invariant sheaf M and $U \subset X$ an open dense subscheme, then in general the restriction morphism $M(X) \to M(U)$ is not injective.

Pure sheaves. We will then call *pure* a sheaf on Sm_S which satisfies the previous property for *any* closed subscheme Z of X of codimension $\geqslant d$ and for which Zariski cohomology agrees with Nisnevich cohomology (see Definition 6.4.9 below).

For instance, if S is normal, any semi-abelian S-scheme $A \to S$ defines a strictly $\mathbb{A}^1$-invariant sheaf which is pure. This follows from the standard properties of abelian schemes [12, Lemma 1] which imply they are flasque sheaves and $\mathbb{A}^1$-invariant. Over a general regular base S, other examples of pure sheaves should be the sheaves associated to the presheaves $X \mapsto H^*_{et}(X; M)$ of étale cohomology with coefficients in a locally constant constructible torsion sheaf M on S of torsion prime to each characteristic of the residue fields of S. But this is not yet known unless S itself is smooth over some base field for instance. It is hard to give other examples of pure sheaves over a general base.

By Lemma 6.4.11 below, over a base field any strictly $\mathbb{A}^1$-invariant sheaf is pure. However over a general base S, it is not true that a strictly

$\mathbb{A}^1$-invariant sheaf on Sm_S is automatically pure. Take a closed immersion $i: Z \subset S$ with Z a non-empty closed subscheme of codimension $d > 0$. Then the (flasque) sheaf $i_*\mathbb{Z}$ is a strictly $\mathbb{A}^1$-invariant sheaf on Sm_S but it is not pure.

We will make in 6.4.12 below a conjecture which implies in particular that for any smooth projective S-scheme X and any integer n the $\mathbb{A}^1$-homotopy sheaf $\pi_n^{\mathbb{A}^1}(X_+)$ is pure. Of course, the assumption that X is projective (and smooth) over S is essential.

Gersten conjecture. By 6.4.15 a strictly $\mathbb{A}^1$-invariant sheaf which is also pure automatically satisfies the Gersten conjecture for all the localizations of points in smooth S-schemes. Thus our Conjectures over a regular base scheme S, together with the representability of algebraic K-theory by the Grassmanian [31] imply Gersten's conjecture for K-groups for all regular local rings. In the same spirit some conjecture of the author predicts that Balmer's Witt groups are represented by a spectrum constructed out of smooth projective S-schemes (orthogonal Grassmanian) so that Gersten conjecture for Witt groups would follow from all these. Finally some version of unramified Milnor's K-theory sheaves [35] over a general regular base scheme S should satisfy Gersten's conjecture as well as they should naturally appear as stable $\mathbb{A}^1$-homotopy sheaves of some explicit algebraic Thom spaces [29].

$\mathbb{A}^1$-homology. For $X \in Sm_S$ let $\mathbb{Z}[X]$ denote the sheaf of abelian groups freely generated by X which we consider as an object in $D_{\mathbb{A}^1}(\mathbb{A}b(S))$; this is the abelianization of the spectrum (X_+) in the sense of Remark 8. For any integer $n \in \mathbb{Z}$ we define the n-th $\mathbb{A}^1$-*homology sheaf*[6] $\mathbb{H}_n^{\mathbb{A}^1}(X)$ of X as to be the n-th homology sheaf of the $\mathbb{A}^1$-localization of $\mathbb{Z}[X]$. As the chain complex $\mathbb{Z}[X]$ is obviously (-1)-connected, if stable $\mathbb{A}^1$-connectivity propertyholds over S, one has $\mathbb{H}_n^{\mathbb{A}^1}(X) = 0$ for $n < 0$. In that case, one can easily deduce that (if the stable $\mathbb{A}^1$-connectivity propertyholds over S) for any $X \in Sm_S$ the canonical Hurewicz morphism (induced by abelianization)

$$\pi_0^{\mathbb{A}^1}(X_+) \to \mathbb{H}_0^{\mathbb{A}^1}(X)$$

is an isomorphism. This result will be strengthened and generalized in [29].

We mention now the following quite natural "topological" conjecture:

CONJECTURE 11. *Assume S is regular.*

(1) *For any $X \in Sm_S$ the sheaves $\mathbb{H}_n^{\mathbb{A}^1}(X)$ vanish for $n < 0$.*
(2) *For any $X \in Sm_S$ of relative dimension $\leqslant d$, the homology sheaves $\mathbb{H}_n^{\mathbb{A}^1}(X)$ vanish for $n > 2d$, and in fact for $n > d$ if X is affine over S.*

[6] As opposed to Suslin–Voevodsky singular homology sheaves $\mathbb{H}_n^S(X)$.

The part (1) of the Conjecture is in fact already implied by Conjecture 2.

The part (2) of the conjecture for all the powers $\mathbb{G}_m^d$ of the multiplicative group over our field k, will be shown in [30] to imply Beilinson–Soulé's vanishing conjecture for all fields extension of k. More generally, that conjecture implies the analogous vanishing for (rational) Suslin singular homology groups.

Our $\mathbb{A}^1$-connectivity results should be thus considered as analogues of the vanishing

$$\pi_n^S(X_+)=0 \quad \text{if } n<0$$

of negative stable homotopy groups of C.W.-complexes. The first type of such a vanishing result is due to Voevodsky [41]. In [29] we will be concerned with the next step: the computation of each of the sheaves $\pi_0^{\mathbb{A}^1}((\mathbb{G}_m)^{\wedge n})=\mathbb{H}_0^{\mathbb{A}^1}((\mathbb{G}_m)^{\wedge n})$ which is the analogue of the computation

$$\pi_0^S(S^0)=\mathbb{Z}$$

In that spirit Conjecture 11 should be thought of as the analogue of the fact that the singular homology groups $H_n(X;\mathbb{Z})$ of a differentiable manifold of dimension d vanish for $n>d$.

The strictly $\mathbb{A}^1$-invariant sheaves will play a central role in our computations, and the study of their basic structures and properties seems to us to be one of the fundamental problems of the subject; this paper and it sequels [29, 30] could be considered as a first small attempt towards the dream of realizing Serre's program [36] in the motivic homotopy theory, which also predicts among other things:

CONJECTURE 12. *Assume S is regular of finite type over $\mathbb{Z}$. For any integer $n\in\mathbb{N}$, any $X\in Sm_S$ the n-th $\mathbb{A}^1$-homology group $\mathbb{H}_n^{\mathbb{A}^1}(X)(S)$ and the n-th stable $\mathbb{A}^1$-homotopy group $\pi_n^{\mathbb{A}^1}(X_+)(S)$ are finite type abelian groups.*

This conjecture seems rather unreachable up to now. We observe that if moreover the stable $\mathbb{A}^1$-connectivity propertyholds over S, by the results of this paper and the representability of algebraic K-theory [31], it implies that Quillen's K-groups are finite type for a regular S of finite type over $\mathbb{Z}$. It also implies the same results for those S for algebraic cobordism groups,[7] and motivic cohomology groups.[8]

Conventions, notations. Everywhere in this paper, S will denote a separated noetherian scheme of finite Krull dimension, Sch_S the category of

[7]Defined over any base using the Thom spectrum $MG\ell$ [41].

[8]For the ones defined over any base by M. Hopkins and the author by "killing" the positive elements of the Lazard in the algebraic cobordism spectrum.

separated S-schemes, and $Sm_S \subset Sch_S$ the full subcategory consisting of finite type smooth S-schemes. We will simply denote by Sm_S the category Sm_S when no confusion can arise.

We will let $Shv(Sm_S)$ (resp. $Shv_\bullet(Sm_S)$, $\mathbb{A}b(Sm_S)$) denote the category of sheaves of sets (resp. of pointed sets, of abelian groups) on Sm_S in the Nisnevich topology [27,31,32]. For any $X \in Sm_S$ the presheaf $Y \mapsto Hom_{Sm_S}(Y, X)$ is a sheaf of sets in the Nisnevich topology, which we call the sheaf represented by X and which we still denote by the letter X. The induced functor

$$Sm_S \to Shv(Sm_S), \quad X \mapsto X$$

is a fully faithful embedding.

By a *point* x on Sm_S, we shall mean a point $x \in X \in Sm_S$. Such a point defines a fiber functor p_x: $Shv(Sm_S) \to \mathcal{S}ets$, $F \mapsto F_x := colim_{V \to X} F(V)$, where the $V \to X$ run over the category of Nisnevich neighborhoods of x, that are étale morphisms $f : V \to X$ such that $f^{-1}(x)$ has only one element with the same residue field of x.

2. Recollection on Simplicial Homotopy Theory

In this section, for the comfort of the reader, we give a brief review of basic notions concerning simplicial sheaves, sheaves of S^1-spectra and the corresponding homotopical algebra.

2.1. SIMPLICIAL SHEAVES

We will assume the reader is familiar with the notion of simplicial objects in a category; see for instance [15,24]. We will also assume the reader is familiar with some of the basic notions of simplicial homotopy theory. We will nevertheless provide a short recollection.

We let $\mathcal{S}$ denote the category of simplicial sets, by $\Delta^{op} Shv(Sm_S)$ that of simplicial sheaves of sets (on Sm_S in the Nisnevich topology) and by $\Delta^{op} Shv_\bullet(Sm_S)$ that of pointed objects in $\Delta^{op} Shv(Sm_S)$, which will be called pointed simplicial sheaves of sets.

For instance, given any set E we still denote by E the sheaf associated to the presheaf $U \mapsto E$ which we call the *constant sheaf* associated to E. Given any simplicial set K, the associated simplicial sheaf of sets on Sm_S is still denoted by K. We thus get a fully faithful embedding $\mathcal{S} \to \Delta^{op} Shv(Sm_S)$, $K \mapsto K$ (if S is assumed to be integral). We observe that this functor commutes to arbitrary small colimits.

For each $n \in \mathbb{N}$, we denote by $\Delta^n \in \mathcal{S}$ the standard simplicial n-simplex. For instance the 0-simplex Δ^0 is also denoted by $*$ and called "the" point as it is the final object in $\mathcal{S}$. We let S^1 denote the quotient in $\mathcal{S}$ of Δ^1 by its boundary $\partial\Delta^1 \subset \Delta^1$, i.e. the disjoint union of its two 0-simplices.

Given two pointed simplicial sheaves $\mathcal{X}$ and $\mathcal{Y}$, we will denote by $\mathcal{X} \vee \mathcal{Y} = X \times \{*\} \cup \{*\} \times Y$ the *wedge* of $\mathcal{X}$ and $\mathcal{Y}$. The wedge is naturally embedded into the product $\mathcal{X} \times \mathcal{Y}$ and the quotient pointed simplicial sheaf of sets $(\mathcal{X} \times \mathcal{Y})/(\mathcal{X} \vee \mathcal{Y})$ is called the *smash-product* of $\mathcal{X}$ and $\mathcal{Y}$ and is denoted by $\mathcal{X} \wedge \mathcal{Y}$.

For a fixed $\mathcal{Y} \in \Delta^{op}Shv(Sm_S)$, the functor $\Delta^{op}Shv(Sm_S) \to \Delta^{op}Shv(Sm_S)$, $\mathcal{X} \mapsto \mathcal{X} \times \mathcal{Y}$ admits a right adjoint $\mathcal{Z} \mapsto \underline{Hom}(\mathcal{Y}, \mathcal{Z})$ and if moreover $\mathcal{Y}$ is pointed then the functor $\Delta^{op}Shv_\bullet(Sm_S) \to \Delta^{op}Shv_\bullet(Sm_S)$, $\mathcal{X} \mapsto \mathcal{X} \wedge \mathcal{Y}$ admits a right adjoint denoted by $\mathcal{Z} \mapsto \underline{Hom}_\bullet(\mathcal{Y}, \mathcal{Z})$. The pointed simplicial sheaf of sets $\underline{Hom}_\bullet(\mathcal{Y}, \mathcal{Z})$ is just the fiber over the base point of $\mathcal{Z}$ of the evaluation at the base point (of $\mathcal{Y}$) morphism $\underline{Hom}(\mathcal{Y}, \mathcal{Z}) \to \mathcal{Z}$.

The *cone* of $\mathcal{X}$ is the pointed simplicial sheaf $C(\mathcal{X}) := \mathcal{X} \wedge \Delta^1$, where Δ^1 is pointed by its 0-vertex $d^1 : * = \Delta^0 \to \Delta^1$. The other 0-vertex $d^0 : * \to \Delta^1$ induces a monomorphism $\mathcal{X} \to C(\mathcal{X})$. The quotient $C(\mathcal{X})/\mathcal{X}$ is isomorphic to the smash-product $\mathcal{X} \wedge S^1$ which is called the *suspension* of $\mathcal{X}$ and is denoted by $\Sigma(\mathcal{X})$. The suspension functor admits as right adjoint the *simplicial loops* space functor

$$\Omega^1 : \Delta^{op}Shv_\bullet(Sm_S) \to \Delta^{op}Shv_\bullet(Sm_S), \quad \mathcal{Z} \mapsto \Omega^1(\mathcal{Z}) := \underline{Hom}_\bullet(S^1, \mathcal{Z})$$

For $f : \mathcal{X} \to \mathcal{Y}$ a morphism of pointed simplicial sheaves we let $C(f)$ denote the amalgamate sum in $Shv_\bullet(Sm_S)$ of the diagram:

$$\begin{array}{ccc} \mathcal{X} & \to & C(\mathcal{X}) \\ \downarrow & & \\ \mathcal{Y} & & \end{array}$$

and call it the *cone* of f. For instance the cone of the morphism $\mathcal{X} \to *$ is by definition the suspension of $\mathcal{X}$.

For any $\mathcal{X} \in \Delta^{op}Shv(Sm_S)$ the equalizer of the diagram in $Shv(Sm_S)$:

$$\mathcal{X}_1 \rightrightarrows \mathcal{X}_0$$

is denoted by $\pi_0(\mathcal{X})$. It is the sheaf associated to the presheaf $U \mapsto \pi_0(\mathcal{X}(U))$.

If we assume moreover that $\mathcal{X} \in \Delta^{op}Shv_\bullet(Sm_S)$, then for any integer $n \geqslant 1$, we shall denote by $\pi_n(\mathcal{X})$ the sheaf of groups (abelian groups if $n \geqslant 2$) associated to the presheaf $U \mapsto \pi_n(\mathcal{X}(U))$. For an integer $n \in \mathbb{N}$ we say that $\mathcal{X}$ is *n-connected* if and only if $\pi_i(\mathcal{X})$ is the trivial sheaf for all $i \in \{0, \ldots, n\}$.

Let x be a point of Sm_S and let $p_x : Shv(Sm_S) \to \mathcal{S}ets$, $F \mapsto F_x$ denote its associated fiber functor. This functor extends to a functor

$$p_x : \Delta^{op}Shv(Sm_S) \to \mathcal{S}, \quad \mathcal{X} \mapsto \mathcal{X}_x$$

If $\mathcal{X}$ is a pointed simplicial sheaf, then clearly one has a canonical isomorphism of groups $\pi_n(\mathcal{X})_x \cong \pi_n(\mathcal{X}_x)$.

DEFINITION 2.1.1. A morphism of simplicial sheaves $f: \mathcal{X} \to \mathcal{Y}$ is called a weak equivalence if and only if for any point x of the site Sm_S the fiber

$$f_x: \mathcal{X}_x \to \mathcal{Y}_x$$

is a weak equivalence of simplicial sets (in the sense of [34]).

The *homotopy category* of simplicial sheaves on Sm_S is the category denoted by $\mathcal{H}_s(Sm_S)$ and obtained from $\Delta^{op}Shv(Sm_S)$ by inverting the weak equivalences.

We may define the notion of weak equivalence between pointed simplicial sheaves to be a morphism $f: \mathcal{X} \to \mathcal{Y}$ in $\Delta^{op}Shv_\bullet(Sm_S)$ whose underlying morphism of simplicial sheaves is a weak equivalence. The *pointed homotopy category* of pointed simplicial sheaves of sets on Sm_S is then the category denoted by $\mathcal{H}_{s,\bullet}(Sm_S)$ and obtained from $\Delta^{op}Shv_\bullet(Sm_S)$ by inverting the weak equivalences.

Remark 2.1.2. In [21], Jardine showed that there is a natural simplicial model category structure on both $\Delta^{op}Shv(Sm_S)$ and $\Delta^{op}Shv_\bullet(Sm_S)$ whose respective homotopy categories are $\mathcal{H}_s(Sm_S)$ and $\mathcal{H}_{s,\bullet}(Sm_S)$. See Appendix A for a quick recollection on simplicial model category structures.

2.2. S^1-SPECTRA

DEFINITION 2.2.1. An S^1-spectrum E in Sm_S is a collection

$$\{E_n, \sigma_n\}_{n \in \mathbb{N}}$$

consisting, for each integer $n \geqslant 0$, of a pointed simplicial sheaf E_n and a morphism $\sigma_n: \Sigma(E_n) = E_n \wedge S^1 \to E_{n+1}$ of pointed simplicial sheaves. Morphisms of S^1-spectra are collections of morphisms of pointed simplicial sheaves which satisfy the obvious conditions. The category of S^1-spectra in Sm_S is denoted by $Sp^{S^1}(Sm_S)$.

EXAMPLE 2.2.2. For any pointed simplicial sheaf $\mathcal{X}$, it suspension spectrum $\Sigma^\infty(\mathcal{X})$ has n-th term $\mathcal{X} \wedge S^n$ (where $S^n := S^1 \wedge \cdots \wedge S^1$) and identities as structure morphisms. This defines the *suspension functor*:

$$\Sigma^\infty: Shv_\bullet(Sm_S) \to Sp^{S^1}(Sm_S)$$

When $\mathcal{X} := *$ is the point, we simply set $S^0 := \Sigma^\infty(*_+)$ and call it the *sphere S^1-spectrum*, or simply the sphere spectrum.

Very often, when no confusion can arise, we will simply denote by $(\mathcal{X})$ the suspension spectrum of a pointed simplicial sheaf $\mathcal{X}$.

EXAMPLE 2.2.3. Let E be an S^1-spectrum and $\mathcal{X}$ be a pointed simplicial sheaf. One defines the smash-product $E \wedge (\mathcal{X})$ as the S^1-spectrum whose n-th term is $E_n \wedge \mathcal{X}$ and with the obvious structure morphisms. This functor induces a functor

$$Sp^{S^1}(Sm_S) \times \Delta^{op} Shv_\bullet(Sm_S) \to Sp^{S^1}(Sm_S), \quad (E, \mathcal{X}) \mapsto E \wedge (\mathcal{X})$$

Observe the formula $S^0 \wedge (\mathcal{X}) = (\mathcal{X})$. For a given $\mathcal{X}$, the above functor admits a right adjoint denoted by $F \mapsto \underline{Hom}_\bullet(\mathcal{X}, F)$, given in degree n by the pointed internal function object $\underline{Hom}_\bullet(\mathcal{X}, F_n)$.

EXAMPLE 2.2.4. The definition of S^1-spectra in Sm_S clearly follows the classical notion of S^1-spectra [1,6]: such an S^1-spectrum E is a collection $\{E_n, \sigma_n\}_{n \in \mathbb{N}}$ consisting, for each integer $n \geqslant 0$, of a pointed simplicial set E_n and a morphism $\sigma_n \colon \Sigma(E_n) = E_n \wedge S^1 \to E_{n+1}$ of pointed simplicial sets. Thus, taking the associated (constant) sheaf defines a functor

$$\to Sp^{S^1}(Sm_S), E \mapsto E$$

from the category of S^1-spectra to that of S^1-spectra in Sm_S.

We observe that for any S^1-spectrum E in Sm_S and any $X \in Sm_S$, the sections on X, $E(X)$, form an S^1-spectrum in. We can thus consider S^1-spectra in $Shv(Sm_S)$ as sheaves on Sm_S of S^1-spectra.

In much the same way, given any point x of Sm_S, the fiber E_x of an S^1-spectrum E at x is an S^1-spectrum in. The fiber functor $p_x \colon \Delta^{op} Shv(Sm_S) \to \mathcal{S}$ thus extends to a functor $p_x \colon Sp^{S^1}(Sm_S) \to$.

DEFINITION 2.2.5. For any S^1-spectrum E, and any $n \in \mathbb{Z}$, the sheaf associated to the presheaf

$$X \mapsto \pi_n(E(X))$$

is denoted $\pi_n(E)$ and is called the n-th homotopy sheaf of E.

Clearly, for any point x of Sm_S, the fiber $\pi_n(E)_x$ at x of $\pi_n(E)$ is canonically isomorphic to the n-homotopy group of the S^1-spectrum E_x.

The n-th homotopy sheaf of E can also be identified to the colimit in the category $\mathbb{A}b(Sm_S)$ of sheaves of abelian groups (for r large enough)

$$\pi_n(E) \cong colim_{r>>0} \pi_{n+r}(E_r)$$

of the diagram whose morphisms $\pi_{n+r}(E_r) \to \pi_{n+r+1}(E_{r+1})$ are induced in the obvious way by the structure morphisms $\sigma_r \colon E_r \wedge S^1 \to E_{r+1}$.

DEFINITION 2.2.6. A morphism of S^1-spectra $f: E \to F$ is called a *stable weak equivalence* if it induces an isomorphism of sheaves

$$\pi_n(E) \cong \pi_n(F)$$

for all integer $n \in \mathbb{Z}$.

As a consequence of our previous comments, a morphism $f: E \to F$ of S^1-spectra is a stable weak equivalence if and only if for any point x of Sm_S, the fiber at x

$$f_x: E_x \to F_x$$

is a stable weak equivalence in.

An S^1-spectrum E whose homotopy groups are all zero is called trivial. This means that the morphism $E \to *$ is a stable weak equivalence.

2.3. STABLE HOMOTOPY THEORY OF S^1-SPECTRA

DEFINITION 2.3.1. The stable homotopy category of S^1-spectra on Sm_S is the category denoted by $\mathcal{SH}_s^{S^1}(Sm_S)$ and obtained from $Sp^{S^1}(Sm_S)$ by inverting the stable weak equivalences. Given S^1-spectra E and F the set of morphisms in $\mathcal{SH}_s^{S^1}(Sm_S)$ between E and F is simply denoted by $[E, F]$.

Remark 2.3.2. The functor $\to$ $Sp^{S^1}(Sm_S)$ (see 2.2.4) clearly maps stable weak equivalences in the usual sense to stable weak equivalences, so that it induces a functor

$$\mathcal{SH} \to \mathcal{SH}_s^{S^1}(Sm_S)$$

where $\mathcal{SH}$ denotes the usual stable homotopy category of [6]. This follows from the obvious fact that the homotopy sheaves associated to $E \in$ are the constant sheaves associated to the homotopy groups of E.

By the very definition, for any point x of Sm_S, the fiber functor at x induces a functor

$$p_x: \mathcal{SH}_s^{S^1}(Sm_S) \to \mathcal{SH}, E \mapsto E_x$$

A brief recollection on the homotopical algebra of spectra [6,22]. One can describe these stable homotopy categories using the “homotopical algebra”of Quillen; see [34] and also Appendix A for a more detailed account. There is a *closed simplicial model category structure* on $Sp^{S^1}(Sm_S)$ whose associated homotopy category is $\mathcal{SH}_s^{S^1}(Sm_S)$. This is proven in [22]. We will only sketch here the concrete consequences concerning the computations in $\mathcal{SH}_s^{S^1}(Sm_S)$.

Given S^1-spectra E and F, a *homotopy* between two morphisms f, $g: E \to F$ is a morphism of S^1-spectra $H: E \wedge (\Delta^1_+) \to F$ such that $H \circ (Id_E \wedge (d^1_+)) = f$ and $H \circ (Id_E \wedge (d^0_+)) = g$. We denote by $\pi(E, F)$ the quotient of $Hom_{Sp^{S^1}(Sm_S)}(E, F)$ by the equivalence relation generated by homotopies. This is called the set of homotopy classes of morphisms from E to F. We observe that the projection $E \wedge (\Delta^1_+) \to E$ is a stable weak equivalence and thus homotopic morphisms induce the same morphism in $\mathcal{SH}^{S^1}_s(Sm_S)$, so that we get a canonical induced map

$$\pi(E, F) \to [E, F]$$

A *homotopy equivalence* $f: E \to F$ is a morphism for which there is a morphism $g: F \to E$ such that $f \circ g$ is homotopic to Id_F and $g \circ f$ is homotopic to Id_E.

A morphism $i: A \to B$ of S^1-spectra is called a *cofibration* if for each $n \geqslant 0$ the morphism of pointed simplicial sheaves $A_n \amalg_{A_{n-1} \wedge S^1} (B_{n-1} \wedge S^1) \to B_n$ is a monomorphism of simplicial sheaves, and call i a *trivial cofibration* if moreover it is a stable weak equivalence. A cofibrant S^1-spectrum E is an S^1-spectrum for which the canonical morphism from the trivial S^1-spectrum $* := \Sigma^\infty(*)$ to E is a cofibration. This exactly means that the structure morphisms σ_n are all monomorphisms of pointed simplicial sheaves. For instance, any suspension spectrum $(\mathcal{X})$ of a pointed simplicial sheaf is cofibrant.

A morphism $f: E \to B$ of S^1-spectra is called a *fibration* if and only if it has the right lifting property[9] with respect to any trivial cofibration. An S^1-spectrum E is called *fibrant* if the morphism $E \to *$ is a fibration, i.e. if for any trivial cofibration $i: A \to B$ and any morphism $f: A \to E$ there is a morphism $g: B \to E$ such that $g \circ i = f$. A *trivial fibration* is a fibration $f: E \to B$ which is also a stable weak equivalence. We observe that a trivial cofibration $F_1 \to F_2$ between fibrant S^1-spectra has to be a simplicial homotopy equivalence and that a trivial fibration between cofibrant S^1-spectra is a homotopy equivalence (use [34]).

The technics from [22] can be easily adapted from presheaves to sheaves to yield:

LEMMA 2.3.3 ([22]). *There exist functors $Sp^{S^1}(Sm_S) \to Sp^{S^1}(Sm_S)$, $E \mapsto E_c$ and $Sp^{S^1}(Sm_S) \to Sp^{S^1}(Sm_S)$, $F \mapsto F_f$ as well as natural transformations (in E and F)*

$$E_c \to E$$

and

$$F \to F_f$$

[9] See Definition A.1.1.

such that for any E*, the spectrum* E_c *is cofibrant and the morphism* $E_c \to E$ *is a trivial fibration and such that for any* F*, the* S^1*-spectrum* F_f *is fibrant and the morphism* $F \to F_f$ *is a trivial cofibration.*

We shall call functors and a natural transformations as in the previous Lemma a *functorial cofibrant resolution* and *functorial fibrant resolution* respectively. We will always assume that such functorial (co-)fibrant resolutions have been chosen in the sequel.

We can then state Quillen's principle of homotopical algebra (see [34] and Appendix A) in the category of S^1-spectra over Sm_S:

LEMMA 2.3.4 ([34]). *Given a cofibrant* S^1*-spectrum* E *and a fibrant* S^1*-spectrum* F *the map*

$$\pi(E, F) \to [E, F]$$

is a bijection.

Given any pair (E, F) of S^1-spectra we can now "compute" the set $[E, F]$ of morphisms in $\mathcal{SH}_s^{S^1}(Sm_S)$ as follows. Since the cofibrant resolution $E_c \to E$ is a stable weak equivalence, the induced map $[E, F] \to [E_c, F]$ is a bijection. In the same way, the map $[E_c, F] \to [E_c, F_f]$ is also a bijection. But now by the previous lemma

$$\pi(E_c, F_f) \cong [E_c, F_f]$$

This is the principle of the *homotopical algebra* of Quillen [34]: to compute $[E, F]^s$ replace E by some cofibrant resolution, replace F by a fibrant resolution and compute the set of homotopy classes between the resolutions.

Remark 2.3.5. The technics of [22] give a model category structure on $Sp^{S^1}(Sm_S)$ (see Appendix A for that notion), but we won't use it in the sequel. The fibrations are not easy to describe (see [6] for instance or [22]). Call a *local fibration* a morphism whose fibers at each point of Sm_S are fibrations in the sense of [6]. Any fibration is a local fibration but the converse is not true; there are local fibrations which are not fibrations. At least we can "describe" *fibrant* S^1-spectra, i.e. those E for which the morphism $E \to *$ is a fibration.

Call a morphism $\mathcal{X} \to \mathcal{Y}$ of simplicial sheaves a *trivial cofibration* if it is both a monomorphism and a weak equivalence. Call a simplicial sheaf $\mathcal{K}$ *fibrant* if for any trivial cofibration $\mathcal{A} \to \mathcal{B}$, any morphism $\mathcal{A} \to \mathcal{K}$ can be extended to $\mathcal{B}$. One then has the following:

LEMMA 2.3.6. *An S^1-spectrum E is fibrant if and only if for each $n \geqslant 0$ the pointed simplicial sheaf E_n is fibrant and the adjoint morphism to σ_n:*

$$\tilde{\sigma_n} : E_n \to \Omega^1(E_{n+1}) = \underline{Hom}_\bullet(S^1, E_{n+1})$$

is a weak equivalence of pointed simplicial sheaves.

Remark 2.3.7. On can also construct the derived category $D(\mathbb{A}b(Sm_S))$ using exactly the same procedure. The principle of homotopical algebra then becomes the principle of "homological algebra", and indeed the fibrant resolution for bounded above complexes correspond to injective resolutions in the usual sense.

The functor

$$Sp^{S^1}(Sm_S) \times \Delta^{op} Shv_\bullet(Sm_S) \to Sp^{S^1}(Sm_S),\ E \mapsto E \wedge (\mathcal{X})$$

preserves stable weak equivalences: this follows easily from the corresponding fact in (see [6]) as well as Lemma 2.2.5. We thus get a functor

$$\mathcal{SH}_s^{S^1}(Sm_S) \times \mathcal{H}_{s,\bullet}(Sm_S) \to \mathcal{SH}_s^{S^1}(Sm_S),\ (E, \mathcal{X}) \mapsto E \wedge (\mathcal{X})$$

Here is an application of the homotopical algebra.

LEMMA 2.3.8. *Let $\mathcal{X}$ be a pointed simplicial sheaf. Then the functor*

$$\mathcal{SH}_s^{S^1}(Sm_S) \to \mathcal{SH}_s^{S^1}(Sm_S),\ E \mapsto E \wedge \mathcal{X}$$

admits as right adjoint the right derived functor[10] *of the functor $F \mapsto \underline{Hom}_\bullet(\mathcal{X}, F)$ which we denote $R\underline{Hom}_\bullet(\mathcal{X}, -)$, and which maps an S^1-spectrum F to:*

$$R\underline{Hom}_\bullet(\mathcal{X}, F) := \underline{Hom}_\bullet(\mathcal{X}, F_f)$$

In the sequel, when no confusion can arise, we shall simply write $F^{(\mathcal{X})}$ instead of $R\underline{Hom}_\bullet(\mathcal{X}, F)$.

Proof. We observe that for any S^1-spectrum E and any S^1-spectrum F we have by adjunction a natural bijection (in E and F)

$$\pi(E \wedge \mathcal{X}, F) \cong \pi(E, \underline{Hom}_\bullet(\mathcal{X}, F))$$

If E is assumed cofibrant and F fibrant, then $E \wedge \mathcal{X}$ is also cofibrant and $\underline{Hom}_\bullet(\mathcal{X}, F)$ is fibrant (easily checked). But then the left hand side can

[10] In the sense of Quillen [34].

be identified to $[E \wedge \mathcal{X}, F]$ and the right hand side to $[E, \underline{Hom}_\bullet(\mathcal{X}, F)]$. This proves that if $f\colon F \to F'$ is a stable weak equivalence between fibrant S^1-spectra, then $\underline{Hom}_\bullet(\mathcal{X}, F) \to \underline{Hom}_\bullet(\mathcal{X}, F')$ is an isomorphism in $\mathcal{SH}_s^{S^1}(Sm_S)$, which is thus a weak equivalence. So that $F \mapsto \underline{Hom}_\bullet(\mathcal{X}, F_f)$ indeed induces a functor $R\underline{Hom}_\bullet(\mathcal{X}, -) : \mathcal{SH}_s^{S^1}(Sm_S) \to \mathcal{SH}_s^{S^1}(Sm_S)$ which is clearly right adjoint to $\mathcal{SH}_s^{S^1}(Sm_S) \to \mathcal{SH}_s^{S^1}(Sm_S)$, $E \mapsto E \wedge \mathcal{X}$ by the previous computation. □

3. The Standard t-Structure for S^1-Spectra

3.1. SUSPENSION AND THE TRIANGULATED STRUCTURE

Our aim is to outline a proof of the following:

PROPOSITION 3.1.1. *The stable homotopy category $\mathcal{SH}_s^{S^1}(Sm_S)$ of S^1-spectra in Sm_S admits a canonical structure of triangulated category in which:*

(1) *the shift functor $E \mapsto E[1]$ is the functor induced by the smash-product by S^1, $E \mapsto E \wedge (S^1)$;*

(2) *an exact triangle is isomorphic to a triangle of the form*

$$E \xrightarrow{f} F \to C(f) \to E[1]$$

where for a morphism of S^1-spectra $f : E \to F$ the spectrum $C(f)$ denotes the cone of f, i.e., the n-th term $C(f)_n$ is precisely the cone of the morphism $f_n : E_n \to F_n$ of pointed simplicial sheaves.

(3) *As a triangulated category, $\mathcal{SH}_s^{S^1}(Sm_S)$ is generated by the objects (U_+), $U \in Sm_S$: this means that a S^1-spectrum E is trivial if and only if for each integer $n \in \mathbb{Z}$ and any $U \in Sm_S$, $[(U_+)[n], E] = 0$;*

(4) *The objects (U_+), $U \in Sm_S$ are "small" in the sense that for any right filtering small category $\mathcal{I}$ and any functor $E_\bullet : \mathcal{I} \to Sp^{S^1}(Sm_S)$, the canonical homomorphism*

$$colim_{i \in \mathcal{I}}[(U_+), E_i] \to [(U_+), colim_{\mathcal{I}} E_\bullet]$$

is an isomorphism.

We first prove:

LEMMA 3.1.2. *The suspension functor*

$$\mathcal{SH}_s^{S^1}(Sm_S) \to \mathcal{SH}_s^{S^1}(Sm_S), E \mapsto E \wedge (S^1)$$

and its right adjoint

$$\mathcal{SH}_s^{S^1}(Sm_S) \to \mathcal{SH}_s^{S^1}(Sm_S),\ E \mapsto E^{(S^1)}$$

are equivalences of categories, inverse to each other.

Proof. Indeed, it suffices to prove that for $E \in Sp^{S^1}(Sm_S)$ the two natural morphisms (coming from the adjunction)

$$E \to (E \wedge S^1)^{(S^1)} \quad \text{and} \quad E^{(S^1)} \wedge S^1 \to E$$

are isomorphisms. We may assume that E is fibrant and the statement follows by the corresponding well-known statement in $\mathcal{SH}$, by Lemma 2.3.6 and by the obvious fact that for any point x of Sm_S

$$(E^{(S^1)})_x \cong (E_x)^{(S^1)}$$

which is easy to check. □

For any integer $n \in \mathbb{Z}$ we will denote by $E \mapsto E[n]$, $\mathcal{SH}_s^{S^1}(Sm_S) \to \mathcal{SH}_s^{S^1}(Sm_S)$ the equivalence of categories $E \mapsto E \wedge ((S^1)^{\wedge n})$ for $n \geqslant 0$ and the functor $E \mapsto E^{(S^1)^{\wedge -n}}$ for $n \leqslant 0$.

Remark 3.1.3. For $U \in Sm_S$ and $n > 0$, one can find a nice S^1-spectrum which is isomorphic to $(U_+)[-n]$; just take as i-term the point $*$ for $i < n$ and $(U_+) \wedge (S^1)^{\wedge i-n}$ for $i \geqslant n$, with the obvious structure morphisms.

LEMMA 3.1.4. *For any S^1-spectrum E, and any $n \in \mathbb{Z}$, the sheaf associated to the presheaf*

$$X \mapsto [(X_+)[n], E]$$

is canonically isomorphic to $\pi_n(E)$.

Proof. One may easily reduce to the case $n = 0$ and E fibrant. Then as (X_+) is a cofibrant S^1-spectrum one gets that $[(X_+)[n], E] \cong \pi((X_+), E)$ which is easily seen to be the same as the set $\pi_0(E_0(X))$. One then concludes using the well-known fact that for any simplicial sheaf $\mathcal{Y}$, the sheaf $\pi_0(\mathcal{Y})$ is the one associated to the presheaf $X \mapsto \pi_0(\mathcal{Y}(X))$. □

We then show that the category $\mathcal{SH}_s^{S^1}(Sm_S)$ is additive. This follows indeed from the previous lemma. Let $\mathcal{H}_\bullet$ denote the homotopy category of pointed simplicial sets. We know that the pointed simplicial circle $S^1 \in \mathcal{H}_\bullet$ has a canonical co-group structure $S^1 \to S^1 \vee S^1$ (corresponding to the fact that the set of morphisms from S^1 to K in $\mathcal{H}_\bullet$ is in one-to-one correspondence with the fundamental group of K). But then the functor

$\mathcal{SH}_s^{S^1}(Sm_S) \times \mathcal{H}_\bullet \to \mathcal{SH}_s^{S^1}(Sm_S), (E, K) \mapsto E \wedge K$ induces a canonical co-group structure in $\mathcal{SH}_s^{S^1}(Sm_S)$ on $E \wedge S^1$ for each E, given by the morphism

$$E \wedge (S^1) \to E \wedge (S^1 \vee S^1) \cong (E \wedge (S^1)) \vee (E \wedge (S^1))$$

By Lemma 3.1.2, we can "cancel" S^1 and we thus get a canonical group structure on any E, which gives the additivity. One can then deduce that finite sums in $\mathcal{SH}_s^{S^1}(Sm_S)$ are also finite products in $\mathcal{SH}_s^{S^1}(Sm_S)$. But in fact one could have proven directly that for two S^1-spectra E and F the morphism of the wedge $E \vee F$ to the categorical product $E \times F$ is an isomorphism because it is so in $\mathcal{SH}$ and using the formulas for any point x of $(Sm_S, \mathcal{T})$

$$E_x \vee F_x \cong (E \vee F)_x \quad \text{and} \quad (E \times F)_x \cong E_x \times F_x$$

Now the fact that the category $\mathcal{SH}_s^{S^1}(Sm_S)$ gets a canonical *triangulated category structure* as in the proposition is "classical". We just mention the proof of:

LEMMA 3.1.5. *Given any morphism $f: E \to F$ of S^1-spectra and an S^1-spectrum G, the cofibration sequence*

$$\cdots \to E \to F \to Cone(f) \to E \wedge S^1 = E[1] \to F[1] \to \ldots$$

induces long exact sequences of abelian groups

$$\cdots \leftarrow [E, G] \leftarrow [F, G] \leftarrow [C(f), G] \leftarrow [E, G[-1]] \leftarrow \ldots$$

and

$$\ldots \to [G, E] \to [G, F] \to [G, C(f)] \to [G, [1]] \to \ldots$$

In particular, one has a long exact sequence of homotopy sheaves

$$\cdots \to \pi_n(E) \to \pi_n(F) \to \pi_n(C(f)) \to \pi_{n-1}(E) \to \ldots$$

Proof. The first long exact sequence is the long homotopy exact sequence of the cofibration sequence as defined by Quillen [34]. The second long exact sequence is an easy consequence of the first one, once one knows that $Z \mapsto Z[1]$ is an equivalence of category. One then deduces the third long exact sequence using Lemma 3.1.4. □

The fact that the (U_+)'s do generate the triangulated category $\mathcal{SH}_s^{S^1}(Sm_S)$ easily follows from Lemma 3.1.4.

It only remains to show that for any $U \in Sm_S$ the suspension spectrum (U_+) is compact. It is not hard to check first that the sets of *homotopy classes* $\pi((U_+), -)$ have the required property that the map

$$colim_{i\in\mathcal{I}}\pi((U_+), E_i) \to \pi((U_+), colim_{\mathcal{I}}E_\bullet)$$

is a bijection, for any right filtering small category $\mathcal{I}$ and any functor $E_\bullet : \mathcal{I} \to Sp^{S^1}(Sm_S)$.

By Lemma 2.3.3 one can pick up a functorial fibrant resolution and thus the natural transformation $E_\bullet \to (E_\bullet)_f$ of functors $\mathcal{I} \to Sp^{S^1}(Sm_S)$ is termwise a stable weak equivalence and moreover for each $i \in \mathcal{I}$ the S^1-spectrum $(E_i)_f$ is fibrant. Thus there is a canonical bijection

$$colim_{i\in\mathcal{I}}[(U_+), E_i] \cong colim_{i\in\mathcal{I}}\pi((U_+), (E_i)_f)$$

Thus if we set $F := colim_{\mathcal{I}}(E_\bullet)_f$), it is sufficient to show that the obvious map

$$\pi((U_+), F) \to [(U_+), F]$$

is bijective.

Unfortunately, in general, the S^1-spectrum $F = colim_{\mathcal{I}}(E_\bullet)_f$ will not be fibrant, though each of the $(E_i)_f$ is. And we can't apply the principle of homotopical algebra.

We will instead use a technic invented by Brown and Gersten [7] in the Zariski topology and adapted to the Nisnevich topology in [31]:

DEFINITION 3.1.6 ([31]). (1) A distinguished square

$$\begin{array}{ccc} W & \to & V \\ \downarrow & & \downarrow \\ U & \to & X \end{array}$$

in Sm_S is a commutative square in which $f : V \to X$ is étale, $U \to X$ is an open immersion, $f^{-1}(U) = W$ and $f^{-1}((X-U)_{red}) \to (X-U)_{red}$ is an isomorphism of schemes. Observe that the morphisms $U \to X$ and $V \to X$ form a Nisnevich covering of X. And, moreover, the Nisnevich topology is generated by the coverings of this form [31, Proposition 1.4 p. 96].

(2) An S^1-spectrum E is a B.G.-S^1-spectrum if for any distinguished square

$$\begin{array}{ccc} W & \to & V \\ \downarrow & & \downarrow \\ U & \to & X \end{array}$$

the square of S^1-spectra

$$\begin{array}{ccc} E(W) & \leftarrow & E(V) \\ \uparrow & & \uparrow \\ E(U) & \leftarrow & E(X) \end{array}$$

is homotopy cartesian.[11]

For instance any fibrant S^1-spectrum is a B.G.-S^1-spectrum, any left filtering colimit of B.G.-S^1-spectra is a B.G.-S^1-spectrum as well.

Now one can easily deduce from [31] the following lemma which finishes the proof of Proposition 3.1.1 because the above spectrum F is also a B.G.-S^1-spectrum:

LEMMA 3.1.7. *For any B.G.-S^1-spectrum E, any $U \in Sm_S$ the canonical map*

$$\pi((U_+), E) \to [(U_+), E]$$

is bijective.

Proof. Indeed the technique of [31] establishes that for any $U \in Sm_S$ the canonical morphism

$$E(U) \to E_f(U)$$

is a stable weak equivalence. But then $\pi((U_+), E) = \pi_0(E(U)) \to \pi_0(E_f(U)) = \pi((U_+), E_f) = [(U_+), E_f] \cong [(U_+), E]$ is a bijection. □

Remark 3.1.8. This can be generalized to presheaves of S^1-spectra (on Sm_S). Let $E: (Sm_S)^{op} \to, U \mapsto E(U)$ be such a presheaf. One says it has the B.G. property if it satisfies the obvious analogue of Definition 3.1.6. Then the technics from [31] indeed yield the statement that for any $U \in Sm_S$ the canonical morphism

$$E(U) \to a(E)_f(U)$$

is a stable weak equivalence, where $a(E)$ denotes the associated sheaf of S^1-spectra.

[11] Or homotopy cocartesian because these notions coincide for spectra.

3.2. EILENBERG–MACLANE SPECTRA AND THE POSTNIKOV TOWER

Recall from [31, p. 57] that for any simplicial sheaf $\mathcal{X}$ is functorialy defined a tower of epimorphisms

$$\{P^n(\mathcal{X})\}_{n\geqslant -1} = \left\{\begin{array}{c} \vdots \\ \downarrow \\ P^n(\mathcal{X}) \\ \downarrow \\ P^{n-1}(\mathcal{X}) \\ \downarrow \\ \vdots \\ P^{-1}(\mathcal{X}) \end{array}\right\}$$

such that for each point x the fiber at x of that tower is exactly the tower of the simplicial set $\mathcal{X}_x$ as constructed in [24, p. 32].

If $\mathcal{X}$ is pointed then $P^{-1}(\mathcal{X}) = *$ is the point but in general $P^{-1}(\mathcal{X})$ is the subsheaf of the point $*$ with fiber at x the empty set if $\mathcal{X}_x = \emptyset$ and the point if $\mathcal{X}_x \neq \emptyset$; this sheaf $P^{-1}(\mathcal{X})$ should be rather denoted $\pi_{-1}(\mathcal{X})$.

When $\mathcal{X}$ is *locally fibrant*, i.e. its fibers are Kan simplicial sets, each of the morphism of this tower is a local fibration and this tower is called the Postnikov tower of $\mathcal{X}$. This is the case for instance when $\mathcal{X}$ is a fibrant simplicial sheaf.

LEMMA 3.2.1. *For any pointed simplicial sheaf $\mathcal{X}$, any natural number $n \geqslant 1$, the canonical morphism*

$$\mathcal{X} \to \Omega^1(P^{n+1}(\mathcal{X} \wedge S^1))$$

factors through $\mathcal{X} \to P^n(\mathcal{X})$ and thus induces a morphism

$$P^n(\mathcal{X}) \to \Omega^1(P^{n+1}(\mathcal{X} \wedge S^1))$$

Proof. This follows formally from the fact that $P^n(\mathcal{X}) \subset \cos k_n(\mathcal{X})$ is the image of $\mathcal{X}$, where $\cos k_n$ is the right adjoint to the n-th skeleton functor sk_n as well as the following easy fact:

$$sk_{n+1}(\mathcal{X} \times \Delta^1) = (sk_n(\mathcal{X}) \times \Delta^1) \bigcup (sk_{n+1}(\mathcal{X}) \times sk_0 \Delta^1)$$

which clearly implies the formula:

$$sk_{n+1}((\mathcal{X}_+) \wedge S^1) = sk_n(\mathcal{X}_+) \wedge S^1$$

□

DEFINITION 3.2.2. (1) Let E be an S^1-spectrum in Sm_S. For any integer $n \in \mathbb{Z}$ we let

$$P^n(E)$$

denote the S^1-spectrum whose r-th term $P^n(E)_r$ is the point $*$ if $n+r \leqslant -1$ and is the pointed simplicial sheaf $P^{n+r}(E_r)$ if $n+r \geqslant 0$, and whose structure morphisms are induced by those of E and the Lemma above.

(2) For any S^1-spectrum E we let

$$\{E_{\leqslant n}\}_{n\in\mathbb{Z}}$$

denote the tower of S^1-spectra in Sm_S, indexed by the integers, whose n-term is $E_{\leqslant n} := P^n(E_f)$. We call this tower the *Postnikov tower* of E. For each $n \in \mathbb{Z}$ we will also denote by $E_{\geqslant n}$ the (homotopy) fiber of the morphism $E_f \to E_{\leqslant n-1}$. We thus have for each $n \in \mathbb{Z}$ an exact triangle in $\mathcal{SH}_s^{S^1}(Sm_S)$ of the form:

$$E_{\geqslant n} \to E \to E_{\leqslant n-1} \to E_{\geqslant n}[1]$$

We observe that for a fibrant S^1-spectrum E the canonical morphism

$$P^n(E) \to E_{\leqslant n} = P^n(E_f)$$

is a stable weak equivalence. Also, it is clear by construction that $\pi_i(E_{\leqslant n}) = 0$ for $i > n$ and that the obvious morphism

$$E \to E_{\leqslant n}$$

induces an isomorphism $\pi_i(E) = \pi_i(E_{\leqslant n})$ for $i \leqslant n$. Thus the homotopy fiber $K_n(E)$ of the morphism $E_{\leqslant n} \to E_{\leqslant n-1}$ is an S^1-spectrum with the property:

$$\pi_i(K_n(E)) = \begin{cases} 0 & \text{if } i \neq n \\ \pi_n(E) & \text{if } i = n \end{cases}$$

For an abelian sheaf $M \in \mathbb{A}b(Sm_S)$ and an integer n recall that the pointed simplicial sheaf $K(M,n)$ (see [31, page 56] for instance) has only one non-trivial homotopy sheaf which is the n-th and is canonically isomorphic to M. This is called the *Eilenberg–MacLane space* of type (M,n). Using the Alexander–Whitney transformation (see [24]) one gets morphisms of pointed simplicial sheaves

$$K(M,n) \wedge S^1 \to K(M,n+1)$$

which turn the collection of $K(M,n)$'s into an S^1-spectrum, which we denote by

$$H(M)$$

Proposition 1.26 of [31] readily implies that this S^1-spectrum has the following property:

LEMMA 3.2.3. *For any $U \in Sm_S$, any integer $n \in \mathbb{Z}$ and any $M \in \mathbb{A}b(Sm_S)$, then the canonical morphism:*

$$H^n_{Nis}(U;M) \to [(U_+), H(M)[n]]$$

is an isomorphism. In particular, of course,

$$\pi_i(H(M)) = \begin{cases} 0 & \text{if } i \neq 0 \\ M & \text{if } i = 0 \end{cases}$$

EXAMPLE 3.2.4. If M is an abelian group, then the cohomology with coefficients in the associated constant sheaf is trivial. For any $X \in Sm_S$, $M(X)$ is the group of locally constant maps from X to M.

Thus $[(X_+)[n], HM[m]] = 0$ unless $n = m$ and then $[(X_+)[n], HM[n]] = M(X)$.

Using Proposition 1.33 of [31], and what we have recalled, one gets:

LEMMA 3.2.5. *Let E be an S^1-spectrum whose homotopy sheaves $\pi_i(E)$ are zero for $i \neq 0$. Then the canonical morphism (in $\mathcal{SH}^{S^1}_s(Sm_S)$)*

$$E \to H(\pi_0(E))$$

is an isomorphism.

As an immediate consequence we get:

COROLLARY 3.2.6. *Let E be an S^1-spectrum and $n \in \mathbb{Z}$ any integer. Then there is a canonical exact triangle in $\mathcal{SH}^{S^1}_s(Sm_S)$ of the form*

$$H(\pi_n(E))[n] \to E_{\leqslant n} \to E_{\leqslant n-1} \to H(\pi_n(E))[n+1]$$

3.3. THE STANDARD t-STRUCTURE AND ITS HEART

DEFINITION 3.3.1. We let $\mathcal{SH}_s^{S^1}(Sm_S)_{\geqslant 0}$ denote the full subcategory of $\mathcal{SH}_s^{S^1}(Sm_S)$ consisting of (-1)-connected (or *connective*) S^1-spectra, i.e. S^1-spectra E with the property that $\pi_i(E)=0$ for all $i<0$.

We let $\mathcal{SH}_s^{S^1}(Sm_S)_{\leqslant 0}$ denote the full subcategory of $\mathcal{SH}_s^{S^1}(Sm_S)$ consisting of S^1-spectra F with the property that $\pi_i(F)=0$ for all $i>0$.

PROPOSITION 3.3.2. *The pair* $(\mathcal{SH}_s^{S^1}(Sm_S)_{\geqslant 0}, \mathcal{SH}_s^{S^1}(Sm_S)_{\leqslant 0})$ *defines a t-structure* [5] *on* $\mathcal{SH}_s^{S^1}(Sm_S)$. *It is non-degenerate in the sense that:*

(1) $\cap_n \mathcal{SH}_s^{S^1}(Sm_S)_{\geqslant 0}=\{0\}$
(2) $\cap_n \mathcal{SH}_s^{S^1}(Sm_S)_{\leqslant 0}=\{0\}$

For any $E\in\mathcal{SH}_s^{S^1}(Sm_S)$, *the exact triangle of Definition 3.2.2:*

$$E_{\geqslant n}\to E\to E_{\leqslant n-1}\to E_{\geqslant n}[1]$$

is exactly the one induced by the t-structure [5].

Moreover the morphisms:

$$hocolim_n E_{\geqslant n}\to E \quad \text{and} \quad E\to holim_n E_{\leqslant n}$$

are both isomorphisms in $\mathcal{SH}_s^{S^1}(Sm_S)$.

This t-structure on $\mathcal{SH}_s^{S^1}(Sm_S)$ *is called the* standard t-structure *on* $\mathcal{SH}_s^{S^1}(Sm_S)$.

The functor $\pi_0: \mathcal{SH}_s^{S^1}(Sm_S)\to \mathbb{A}b(Sm_S)$, $E\mapsto \pi_0(E)$ *induces an equivalence of abelian categories between the* heart[12] *of the standard t-structure and the category of abelian sheaves on* Sm_S, *whose inverse is the functor* $H: \mathbb{A}b(Sm_S)\to\mathcal{SH}_s^{S^1}(Sm_S)$, $M\mapsto H(M)$.

This proposition follows rather clearly from what we have done so far as well as the following two lemmas:

LEMMA 3.3.3. *Let E be an S^1-spectrum, $n\in\mathbb{Z}$ and $U\in Sm_S$ of Krull dimension d. Then for any $i\in\mathbb{Z}$ the morphism*

$$[(U_+)[i], E]\to[(U_+)[i], E_{\leqslant n}]$$

is onto for $n\geqslant i+d-1$ and an isomorphism for $n\geqslant i+d$.

Proof. *This is clearly implied by Corollary 1.41 of [31].* □

LEMMA 3.3.4. *Let E be an S^1-spectrum on Sm_S. The following conditions are then equivalent:*

[12] i.e. the intersection $\mathcal{SH}_s^{S^1}(Sm_S)_{\geqslant 0}\cap\mathcal{SH}_s^{S^1}(Sm_S)_{\leqslant 0}$, which is abelian [5].

(i) $E \in \mathcal{SH}_s^{S^1}(Sm_S)_{\leqslant -1}$;
(ii) for any $U \in Sm_S$ *and any* $n \leqslant 0$ *the group* $[(U_+)[n], E]$ *vanishes.*

The following conditions are also equivalent:

(i) E *is* (-1)*-connected;*
(ii) E *is isomorphic in* $\mathcal{SH}_s^{S^1}(Sm_S)$ *to the telescope of a diagram:*

$$* = E^0 \to \cdots \to E^n \to \ldots$$

with E^n *the cone of a morphism of spectra*

$$\vee_\alpha((X_\alpha)_+)[n_\alpha - 1] \to E^{n-1}$$

where the α*'s run in some set* I_n*, with* $X_\alpha \in Sm_S$ *and* $n_\alpha \geqslant 0$.

Proof. This easily follows from the fact that $\mathcal{SH}_s^{S^1}(Sm_S)$ is generated (as a triangulated category) by the (U_+)'s 3.1.1, Proposition 3.1.1 4) and from Lemma 3.1.4. □

Remark 3.3.5. Using the previous results one easily sees that, when the base S is irreducible, the functor $\mathcal{SH} \to \mathcal{SH}_s^{S^1}(Sm_S)$ of 2.3.2 is a fully faithful embedding and preserves the standard t-structures. Indeed, by Example 3.2.4, $[S^0[n], HM[m]] = 0$ unless $n = m$ and then $[S^0[n], HM[n]] = M$ for constant abelian sheaf M. Then one concludes using Postnikov tower on the target and skeletal filtration on the source for classical spectra.

4. Stable $\mathbb{A}^1$-Homotopy Theory of S^1-Spectra

4.1. $\mathbb{A}^1$-LOCAL S^1-SPECTRA AND STABLE $\mathbb{A}^1$-WEAK EQUIVALENCES

Recall that for a simplicial sheaf $\mathcal{X}$, $\mathcal{X}_+$ denotes the pointed simplicial sheaf obtained from $\mathcal{X}$ by adding a disjoint base point.

DEFINITION 4.1.1. (1) An S^1-spectrum $E \in Sp^{S^1}(Sm_S)$ is called $\mathbb{A}^1$-*local* if and only if for any $F \in Sp^{S^1}(Sm_S)$, the projection $F \wedge (\mathbb{A}^1_+) \to F$ induces an isomorphism of abelian groups:

$$[F, E] \to [F \wedge (\mathbb{A}^1_+), E]$$

We shall denote by $\mathcal{SH}_{\mathbb{A}^1-loc}^{S^1}(Sm_S)$ the full subcategory of $\mathcal{SH}_s^{S^1}(Sm_S)$ consisting of $\mathbb{A}^1$-local S^1-spectra.

(2) A morphism $f: X \to Y$ in $Sp^{S^1}(Sm_S)$ is called a *stable* $\mathbb{A}^1$-*weak equivalence* if and only if for any $\mathbb{A}^1$-local spectra E, the map:

$$[Y, E] \to [X, E]$$

is an isomorphism.

(3) The stable $\mathbb{A}^1$-homotopy category of S^1-spectra is the one obtained from $Sp^{S^1}(Sm_S)$ by inverting the stable $\mathbb{A}^1$-weak equivalences and is denoted by $\mathcal{SH}^{S^1}_{\mathbb{A}^1}(Sm_S)$.

Observe that for any $F \in Sp^{S^1}(Sm_S)$ the morphism $F \wedge (\mathbb{A}^1_+) \to F$ is a stable $\mathbb{A}^1$-weak equivalence by definition. Also any stable weak equivalence is a stable $\mathbb{A}^1$-weak equivalence. Of course the category $\mathcal{SH}^{S^1}_{\mathbb{A}^1}(Sm_S)$ can also be considered as obtained from $\mathcal{SH}^{S^1}_s(Sm_S)$ by inverting the stable $\mathbb{A}^1$-weak equivalences. Observe also that for a given pointed simplicial sheaf $\mathcal{X}$, the functor $\mathcal{SH}^{S^1}_s(Sm_S) \to \mathcal{SH}^{S^1}_s(Sm_S)$, $E \mapsto E \wedge (\mathcal{X})$ preserves stable $\mathbb{A}^1$-weak equivalences (use Lemma 2.3.8), and thus induces a functor $\mathcal{SH}^{S^1}_{\mathbb{A}^1}(Sm_S) \to \mathcal{SH}^{S^1}_{\mathbb{A}^1}(Sm_S)$ still denoted by $E \mapsto E \wedge (\mathcal{X})$. For instance the shift functor on $\mathcal{SH}^{S^1}_s(Sm_S)$ induces a shift functor on $\mathcal{SH}^{S^1}_{\mathbb{A}^1}(Sm_S)$, denoted by $E \mapsto E[1]$.

In the following lemma, we give several equivalent conditions for an S^1-spectrum to be $\mathbb{A}^1$-local. When needed, we always consider the affine line $\mathbb{A}^1$ as pointed by its 0-section.

LEMMA 4.1.2. *Let E be a S^1-spectrum. Then the following conditions are equivalent:*

(i) *E is $\mathbb{A}^1$-local;*
(ii) *The obvious morphism $E \to E^{(\mathbb{A}^1_+)}$ is a stable weak equivalence;*
(iii) *The functional object $E^{(\mathbb{A}^1)}$ is trivial;*
(iv) *For any $U \in Sm_S$, any integer $n \in \mathbb{Z}$ the homomorphism*

$$[(U_+)[n], E] \to [((U \times \mathbb{A}^1)_+)[n], E]$$

is an isomorphism;
(v) *For any $U \in Sm_S$, any integer $n \in \mathbb{Z}$, the "evaluation at 1" morphism*

$$ev_1 : [(U_+)[n], E^{(\mathbb{A}^1)}] \to [(U_+)[n], E]$$

is the zero map.

Proof. The equivalence $(i) \Leftrightarrow (ii)$ obvious in view of Lemma 2.3.8.

From the exact triangle

$$S^0 := *_+ \to \mathbb{A}^1_+ \to \mathbb{A}^1$$

(in which $S^0 \to \mathbb{A}^1_+$ maps the non-base point to 0) we get an exact triangle

$$E^{(\mathbb{A}^1)} \to E^{(\mathbb{A}^1_+)} \to E$$

which proves the equivalence (ii) $\Leftrightarrow$ (iii) because the evaluation at 0, $E^{(\mathbb{A}^1_+)} \to E$ which appears is a left inverse to the morphism $E \to E^{(\mathbb{A}^1_+)}$.

The equivalence (ii)⇔(iv) clearly follows from the fact that the spectra (U_+) generate the triangulated category $\mathcal{SH}_s^{S^1}(Sm_S)$ 3.1.1.
Using this we see that (iii) is equivalent to:

(iii)': For any $U \in Sm_S$, any integer $n \in \mathbb{N}$, the group $[(U_+)[n] \wedge \mathbb{A}^1, E]$ vanishes.

The implication $(\text{iii})' \Rightarrow (v)$ is trivial (because the group $[(U_+)[n], E^{(\mathbb{A}^1)}]$ becomes trivial). Let's prove the converse implication $(\text{v}) \Rightarrow (\text{iii})'$. Assume (v) and fix $U \in Sm_S$ and $n \in \mathbb{Z}$. We want to show that any $\mathcal{SH}_s^{S^1}(Sm_S)$-morphism $f: (U_+)[n] \wedge \mathbb{A}^1 \to E$ is trivial.

But for any morphism of S^1-spectra

$$f: F \wedge (\mathbb{A}^1) \to E$$

let $\tilde{f}: F \wedge \mathbb{A}^1 \to E^{(\mathbb{A}^1)}$ be the adjoint of the composition

$$F \wedge (\mathbb{A}^1 \wedge \mathbb{A}^1) \overset{Id_F \wedge \mu}{\to} F \wedge \mathbb{A}^1 \overset{f}{\to} E$$

where $\mu: \mathbb{A}^1 \wedge \mathbb{A}^1 \to \mathbb{A}^1$ denote (the morphism of sheaves induced by) the product of the ringed object $\mathbb{A}^1$. Then the following diagram is commutative (in $Sp^{S^1}(Sm_S)$):

$$\begin{array}{ccc} F \wedge \mathbb{A}^1 & \overset{\tilde{f}}{\to} & E^{(\mathbb{A}^1)} \\ || & & \downarrow ev_1 \\ F \wedge \mathbb{A}^1 & \overset{f}{\to} & E \end{array}$$

This fact (applied with $F = \Sigma^\infty(U_+)[n]$ clearly implies the claim. □

Remark 4.1.3. Using the terminology of [31] we can also observe that a fibrant S^1-spectrum E is $\mathbb{A}^1$-local if and only if:
(vi): each term E_n is an $\mathbb{A}^1$-local simplicial sheaf.

Lemma 3.1.7 clearly implies:

LEMMA 4.1.4. *Let E be any B.G.-S^1-spectrum. Then it is $\mathbb{A}^1$-local if and only if it is $\mathbb{A}^1$-invariant in the sense that for any $U \in Sm_S$ the canonical morphism $E(U) \to E(U \times \mathbb{A}^1)$ is a stable weak equivalence.*

Moreover, in that case, for any $U \in Sm_S$, any $n \in \mathbb{Z}$ the canonical map

$$\pi_n(E(U)) \to [(U_+)[n], E]_{\mathbb{A}^1}$$

is an isomorphism.

Remark 4.1.5. Together with V. Voevodsky, we studied in [31] the analogous "unstable" notions in $\Delta^{op}Shv(Sm_S)$. A simplicial sheaf $\mathcal{X}$ is $\mathbb{A}^1$-*local* if and only if any $\mathcal{Y} \in \Delta^{op} Shv(Sm_S)$, the projection $\mathcal{Y} \times \mathbb{A}^1 \to \mathcal{Y}$ induces a bijection:

$$Hom_{\mathcal{H}_s(Sm_S)}(\mathcal{Y}, \mathcal{X}) \to Hom_{\mathcal{H}_s(Sm_S)}(\mathcal{Y} \times \mathbb{A}^1, \mathcal{X})$$

thus defining the full subcategory $\mathcal{H}_{\mathbb{A}^1}(Sm_{S\mathcal{T}}) \subset \mathcal{H}_s(Sm_S)$ consisting of $\mathbb{A}^1$-local simplicial sheaves. A morphism $f: \mathcal{X} \to \mathcal{Y}$ of simplicial sheaves is an $\mathbb{A}^1$-*weak equivalence* if and only if for any $\mathbb{A}^1$-local $\mathcal{Z}$ the map

$$Hom_{\mathcal{H}_s(Sm_S)}(\mathcal{Y}, \mathcal{Z}) \to Hom_{\mathcal{H}_s(Sm_S)}(\mathcal{X}, \mathcal{Z})$$

is bijective. The category obtained from $\mathcal{H}_s(Sm_S)$ by formally inverting the $\mathbb{A}^1$-weak equivalences is called the $\mathbb{A}^1$-*homotopy category* of Sm_S.

We proved there that the inclusion $\mathcal{H}_{\mathbb{A}^1}(Sm_{S\mathcal{T}}) \subset \mathcal{H}_s(Sm_S)$ admits a left adjoint

$$L_{\mathbb{A}^1}^{unst}: \mathcal{H}_s(Sm_S) \to \mathcal{H}_{\mathbb{A}^1}(Sm_{S\mathcal{T}})$$

called the $\mathbb{A}^1$-localization functor. Its existence has the formal consequence that the functor $L_{\mathbb{A}^1}^{unst}$ induces an equivalence between that category and $\mathcal{H}_{\mathbb{A}^1}(Sm_{S\mathcal{T}})$. In the next section we do the same thing, in a slightly more convenient way for us, for the stable homotopy category $\mathcal{SH}_s^{S^1}(Sm_S)$.

4.2. $\mathbb{A}^1$-LOCALIZATION

THEOREM 4.2.1. *The inclusion*

$$\mathcal{SH}_{\mathbb{A}^1-loc}^{S^1}(Sm_S) \subset \mathcal{SH}_s^{S^1}(Sm_S)$$

admits a left adjoint

$$L_{\mathbb{A}^1}: \mathcal{SH}_s^{S^1}(Sm_S) \to \mathcal{SH}_{\mathbb{A}^1-loc}^{S^1}(Sm_S)$$

As a consequence, a morphism f in $\mathcal{SH}_s^{S^1}(Sm_S)$ is a stable $\mathbb{A}^1$-weak equivalence if and only if $L_{\mathbb{A}^1}(f)$ is an isomorphism (i.e. a stable weak equivalence in $Sp^{S^1}(Sm_S)$).

Recall that $\mathcal{SH}_{\mathbb{A}^1}^{S^1}(Sm_S)$ is the category obtained from $\mathcal{SH}_s^{S^1}(Sm_S)$ by formally inverting stable $\mathbb{A}^1$-weak equivalences. For two S^1-spectra E and F we shall denote by

$$[E, F]_{\mathbb{A}^1}$$

the set of morphisms $Hom_{\mathcal{SH}_{\mathbb{A}^1}^{S^1}(Sm_S)}(E, F)$.

DEFINITION 4.2.2. Let $E \in Sp^{S^1}(Sm_S)$ and $n \in \mathbb{Z}$. We set

$$\pi_n^{\mathbb{A}^1}(E) := \pi_n(L_{\mathbb{A}^1}(E)) \in \mathbb{A}b(Sm_S)$$

and call it the n-th A^1-homotopy sheaf of E.

Of course, $\pi_n^{\mathbb{A}^1}(E)$ is also the associated sheaf to $X \mapsto [(X_+)[n], E]_{\mathbb{A}^1}$.
It is quite formal to deduce the following corollary from the theorem.

COROLLARY 4.2.3.
(1) *The functor* $L_{\mathbb{A}^1}$ *induces an equivalence of categories*

$$L_{\mathbb{A}^1}: \mathcal{SH}_{\mathbb{A}^1}^{S^1}(Sm_S) \cong \mathcal{SH}_{\mathbb{A}^1-loc}^{S^1}(Sm_S)$$

and the induced functor $\mathcal{SH}_{\mathbb{A}^1}^{S^1}(Sm_S) \to \mathcal{SH}_s^{S^1}(Sm_S)$ *is left adjoint to* $\mathcal{SH}_s^{S^1}(Sm_S) \to \mathcal{SH}_{\mathbb{A}^1}^{S^1}(Sm_S)$.
(2) *The category* $\mathcal{SH}_{\mathbb{A}^1}^{S^1}(Sm_S)$ *admits a unique triangulated category structure whose shift is* $E \mapsto E[1]$ *and which turns the functors* $\mathcal{SH}_s^{S^1}(Sm_S) \to \mathcal{SH}_{\mathbb{A}^1}^{S^1}(Sm_S)$ *and* $\mathcal{SH}_{\mathbb{A}^1}^{S^1}(Sm_S) \cong \mathcal{SH}_{\mathbb{A}^1-loc}^{S^1}(Sm_S) \subset \mathcal{SH}_s^{S^1}(Sm_S)$ *into exact functors. In particular the* $\mathbb{A}^1$*-localization functor* $\mathcal{SH}_s^{S^1}(Sm_S) \to \mathcal{SH}_{\mathbb{A}^1-loc}^{S^1}(Sm_S)$ *preserves exact triangles.*
(3) *For two* S^1*-spectra* E *and* F, *the group* $[E, F]_{\mathbb{A}^1}$ *can be identified with* $[L_{\mathbb{A}^1}(E), L_{\mathbb{A}^1}(F)] = [E, L_{\mathbb{A}^1}(F)]$.

Proof of the Theorem. In this proof we always consider the affine line $\mathbb{A}^1$ as pointed by the zero section $0: S \to \mathbb{A}^1$. For any $E \in Sp^{S^1}(Sm_S)$, let's denote by $ev_1: E^{(\mathbb{A}^1)} \to E$ the evaluation morphism at 1.

We first prove (1). Let $E \mapsto E_f$ be a fixed functorial fibrant resolution. For any endofunctor $\mathcal{F}: Sp^{S^1}(Sm_S) \to Sp^{S^1}(Sm_S)$ we let $\mathcal{F}_f$ be the functor $E \mapsto \mathcal{F}(E)_f$.

Fix an S^1-spectrum E. We define $L^{(1)}(E)$ as the cone of the obvious morphism $ev_1: E_f^{(\mathbb{A}^1)} \to E_f$. Let $E \to L_f^{(1)}(E)$ be the obvious morphism. Define by induction on $n \geqslant 1$, the functor $L^{(n)} := L_f^{(1)} \circ L_f^{(n-1)}$. We have natural morphisms $L_f^{(n-1)}(E) \to L_f^{(n)}(E)$ and we let $L^{\infty}(E) = Tel_{n \in \mathbb{N}} L_f^{(n)}(E)$ be the colimit of this diagram.

We claim that the S^1-spectrum $L^{\infty}(E)$ is $\mathbb{A}^1$-local and that the morphism

$$E \to L^{\infty}(E)$$

is a stable $\mathbb{A}^1$-weak equivalence, which proves the theorem.

To do so, use (i)$\Leftrightarrow$(iv) of Lemma 4.1.2 and Proposition 3.1.1. (4). This is an analogue of the "small object argument" of Quillen [34] which uses Proposition 3.1.1 (4).

Moreover we observe that $L_f^{(n-1)}(E) \to L_f^{(n)}(E)$ is always an $\mathbb{A}^1$-weak equivalence, proving that so is $E \to L^\infty(E)$. Indeed the fiber of this morphism is stably weakly equivalent to $(E_f)^{\mathbb{A}^1}$ by construction; but the morphism $(E_f)^{\mathbb{A}^1} \wedge \mathbb{A}^1 \to (E_f)^{\mathbb{A}^1}$ which is adjoint to the morphism $(E_f)^{\mathbb{A}^1} \to (E_f)^{\mathbb{A}^1 \wedge \mathbb{A}^1}$ (induced by the product $\mu: \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$) is left inverse to the morphism $(E_f)^{\mathbb{A}^1} \to (E_f)^{\mathbb{A}^1} \wedge \mathbb{A}^1$ (induced by $S^0 \to \mathbb{A}^1$), proving that $(E_f)^{\mathbb{A}^1}$ is a direct factor of $(E_f)^{\mathbb{A}^1} \wedge \mathbb{A}^1$ and thus that $(E_f)^{\mathbb{A}^1} \to *$ is a stable $\mathbb{A}^1$-weak equivalence. The morphism $L_f^{(n-1)}(E) \to L_f^{(n)}(E)$ is thus a stable $\mathbb{A}^1$-weak equivalence as well.

The exactness of $E \to L^\infty(E)$ is easy to check. □

LEMMA 4.2.4.
(1) *A wedge of* $\mathbb{A}^1$*-local* S^1*-spectra is still* $\mathbb{A}^1$*-local.*
(2) *The* $\mathbb{A}^1$*-localization of a wedge of* S^1*-spectra is the wedge of the corresponding* $\mathbb{A}^1$*-localizations.*

Proof. (1) is clear by Proposition 3.1.1. (2) follows from the easy fact that a direct sum of stable $\mathbb{A}^1$-weak equivalences is a stable $\mathbb{A}^1$-weak equivalence (map it to an $\mathbb{A}^1$-local spectrum). □

Remark 4.2.5. The lemma implies that the $\mathbb{A}^1$-localization functor commutes to countable telescopes as well. It is possible to deduce from that fact a simplicial model category structure on $Sp^{S^1}(Sm_S)$ in which weak equivalences are stable $\mathbb{A}^1$-weak equivalences and cofibrations are the same as in the simplicial model structure.

Remark 4.2.6. As in Remark 3.1.8, Lemma 4.1.4 can be generalized to presheaves of S^1-spectra (on Sm_S). Let $E: (Sm_S)^{op} \to, U \mapsto E(U)$ be such a presheaf which has the B.G.-property and the $\mathbb{A}^1$-invariance property. Then for any $U \in Sm_S$ the canonical morphism

$$E(U) \to L_{\mathbb{A}^1}(a(E))(U) = [(U_+), a(E)]_{\mathbb{A}^1}$$

is a stable weak equivalence, where $a(E)$ denotes the associated sheaf of S^1-spectra.

COROLLARY 4.2.7. *The objects* (U_+), $U \in Sm_S$ *are "small" in* $\mathcal{SH}^{S^1}_{\mathbb{A}^1}(Sm_S)$ *in the sense that for any right filtering small category* $\mathcal{I}$ *and any functor* $E_\bullet: \mathcal{I} \to Sp^{S^1}(Sm_S)$*, the canonical homomorphism*

$$colim_{i \in \mathcal{I}}[(U_+), E_i]_{\mathbb{A}^1} \to [(U_+), colim_{\mathcal{I}} E_\bullet]_{\mathbb{A}^1}$$

is an isomorphism.

Proof. Proposition 3.1.1 (4) implies that

$$colim_{i\in\mathcal{I}}[(U_+), E_i]_{\mathbb{A}^1} \cong [(U_+), colim_{\mathcal{I}} L_{\mathbb{A}^1}(E_\bullet)]$$

It is clear that the S^1-spectrum $colim_{\mathcal{I}} L_{\mathbb{A}^1}(E_\bullet)$ has the B.G.-property and the $\mathbb{A}^1$-invariance property, thus it is $\mathbb{A}^1$-local by Lemma 4.1.4. Now, filtering colimits of $\mathbb{A}^1$-weak equivalences are $\mathbb{A}^1$-weak equivalences by [31, Corollary 2.13 p. 73] and thus the morphism of S^1-spectra $colim_{\mathcal{I}} E_\bullet \to colim_{\mathcal{I}} L_{\mathbb{A}^1}(E_\bullet)$ is a stable $\mathbb{A}^1$-weak equivalence to an $\mathbb{A}^1$-local S^1-spectrum, it is thus an $\mathbb{A}^1$-localization. Thus $[(U_+), colim_{\mathcal{I}} E_\bullet]_{\mathbb{A}^1} \cong [(U_+), colim_{\mathcal{I}} L_{\mathbb{A}^1}(E_\bullet)]$, which finishes the proof. □

We know describe our construction of the localization functor in a slightly different way. Denote by C the quotient sheaf of pointed sets $\mathbb{A}^1/(Spec(k)_+)$ where the closed immersion $Spec(k)_+ \to \mathbb{A}^1$ maps the base point to 0 and $Spec(k)$ to 1. We thus get an exact triangle of S^1-spectra

$$S^0 = (Spec(k)_+) \to (\mathbb{A}^1) \to (C) \to S^0[1]$$

Set $\sigma := (C)[-1]$ so that we have now an exact triangle of the form

$$(\mathcal{T}) \quad \sigma \to S^0 \to (\mathbb{A}^1) \to \sigma[1]$$

For any S^1-spectrum E set $E^\sigma := (E^{(C)})[1]$; one has an exact triangle of the form:

$$E^{(\mathbb{A}^1)} \to E \to E^\sigma$$

obtained by mapping $(\mathcal{T})$ into E. Thus in fact, the S^1-spectrum E^σ is canonically isomorphic in $\mathcal{SH}_s^{S^1}(Sm_S)$ to $L_f^{(1)}(E)$. Iterating this procedure, for any integer $m \geqslant 0$ we let $\sigma^{\wedge m}$ denote $(C^{\wedge m})[-m]$ and we let $E^{\sigma^{\wedge m}}$ denote $(E^{(C^{\wedge m})})[m]$. The map $\sigma \to S^0$ induces morphisms

$$E^{\sigma^{\wedge(m-1)}} \to E^{\sigma^{\wedge m}}$$

and clearly $E^{\sigma^{\wedge m}}$ is canonically isomorphic to $L_f^{(m)}(E)$ in such a way $L_{\mathbb{A}^1}(E)$ can be identified to the telescope of the diagram

$$E \to E^\sigma \to \cdots \to E^{\sigma^{\wedge m}} \to \dots$$

This description will be quite useful in the next section.

4.3. A VANISHING RESULT

LEMMA 4.3.1. ([41, 4.14]). *Assume $X \in Sm_S$ has Krull dimension d. Let $n \in \mathbb{Z}$ and let E be an connective S^1-spectrum. Then the group*

$$[(X_+), L_{\mathbb{A}^1}(E)[n]] = [(X_+), E[n]]_{\mathbb{A}^1}$$

vanishes for $n > d$.

Proof. By the construction we gave above of the $\mathbb{A}^1$-localization functor and Proposition 3.1.1 (4) the morphism

$$colim_m[(X_+), E^{\sigma^{\wedge m}}[n]] \to [(X_+), L_{\mathbb{A}^1}(E)[n]]$$

is an isomorphism. Thus it suffices to prove that for each m the group $[(X_+), E^{\sigma^{\wedge m}}[n]]$ vanishes. It is clear that for any S^1-spectrum F, the group $[F, E^{\sigma^{\wedge m}}]$ is canonically isomorphic to $[F \wedge C^{\wedge m}[-m], E]$. Thus it suffices to prove that the group $[(X_+) \wedge C^{\wedge m}, E[n+m]]$ vanishes for $n > d$. Using the Postnikov tower of E and the Lemma 3.3.3 we end up with proving the vanishing of the groups $[(X_+) \wedge C^{\wedge m}, H(M)[n]]$ for $n > d+m$ for any $M \in \mathbb{A}b(Sm_S)$. This is proven in Lemma 4.3.2 below. □

LEMMA 4.3.2. *Assume $X \in Sm_S$ is of Krull dimension d. Let $n \in \mathbb{Z}$ and let E be a connective S^1-spectrum. Then the group*

$$[(X_+) \wedge C^{\wedge m}, E[n]]$$

vanishes for $n > d+m$.

Proof. We prove by induction on $m \geqslant 0$ that for any $i \in \mathbb{N}$ the group

$$[(X_+) \wedge C^{\wedge m} \wedge (\mathbb{A}^1)^{\wedge i}, E[n]]$$

is trivial for $n > d+m+i$.

This is true for $m = 0$ because $X \times \mathbb{A}^i$ is of Krull dimension $d+i$, because the Nisnevich cohomological dimension of a scheme is less or equal to its Krull dimension, because the S^1-spectrum $(X_+) \wedge (\mathbb{A}^1)^{\wedge i}$ is a direct factor of $((X \times \mathbb{A}^i)_+)$ and because of Lemma 3.2.3.

Assume now the inductive hypothesis for $m-1 \geqslant 0$. Let $i \in \mathbb{N}$ and $n > d+m+i$. Using the triangle which defines C we get an exact sequence of groups

$$[(X_+) \wedge C^{\wedge(m-1)} \wedge (\mathbb{A}^1)^{\wedge i}, E[n-1]] \to \left[(X_+) \wedge C^{\wedge m} \wedge (\mathbb{A}^1)^{\wedge i}, E[n]\right]$$
$$\to [(X_+) \wedge C^{\wedge(m_1)} \wedge (\mathbb{A}^1)^{\wedge(i+1)}, E[n]]$$

which easily implies the result by induction. □

COROLLARY 4.3.3. *Assume $S = Speck$ is the spectrum of a field and let $X \in Sm_S$ be a 0-dimensional scheme. Let $n < 0$ be an integer and let E be a (-1)-connected S^1-spectrum on Sm_S. Then the group*

$$[(X_+)[n], L_{\mathbb{A}^1}(E)] = [(X_+), E[n]]_{\mathbb{A}^1} = \pi_n^{\mathbb{A}^1}(E)(X)$$

vanishes. For instance

$$[S^0[n], L_{\mathbb{A}^1}(S^0)] = [S^0[n], S^0]_{\mathbb{A}^1} = \pi_n^{\mathbb{A}^1}(S^0)(k) = 0$$

The last equality is shown using the Postnikov tower of $L_{\mathbb{A}^1}(E)$.

5. Base Change and Gluing

5.1. BASE CHANGE

Let $f\colon S' \to S$ be a morphism with S' irreducible, separated, noetherian of finite Krull dimension.

We denote by $Sm_S{}'$ the category $Sm_{S'}$ of smooth finite type S'-schemes and by

$$f^{-1}\colon Sm_S \to Sm_S{}'$$

the obvious functor $(X \to S) \mapsto (X \times_S S')$. We endow $Sm_S{}'$ with the Nisnevich topology so that for any sheaf $F \in Shv(Sm_S{}')$ the composition $F \circ f^{-1}$ is a sheaf on $Sm_{S\mathcal{T}}$. We denote by

$$f_*\colon Shv(Sm_S{}') \to Shv(Sm_S)$$

the functor so obtained. It admits a left adjoint

$$f^*\colon Shv(Sm_S) \to Shv(Sm_S{}')$$

with the property that it maps the sheaf represented by $X \in Sm_S$, to the sheaf $f^*(X)$ represented by $f^{-1}(X) \in Sm_S{}'$. This pair of adjoint functors extends to a pair of adjoint functors

$$f_*\colon Sp^{S^1}(Sm_S{}') \to Sp^{S^1}(Sm_S)$$

defined by the formula $f_*(E)(U) := E(f^{-1}(U)) \in$ for the right adjoint and

$$f^*\colon Sp^{S^1}(Sm_S) \to Sp^{S^1}(Sm_S{}')$$

which maps $E \in Sp^{S^1}(Sm_S)$ to the S^1-spectrum $f^*(E)$ in $Sm_S{}'$ with n-term $f^*(E_n)$ and structure morphisms defined using the fact that $f^*(E_n) \wedge S^1 = f^*(E_n) \wedge f^*(S^1) \cong f^*(E_n \wedge S^1)$: this follows trivially from the fact that f^* does commute to sums. We observe in particular the formula $f^*((U_+)) = (f^{-1}(U)_+)$ for $U \in Sm_S$.

Base change with respect to smooth morphism. Assume now that the morphism $f\colon S' \to S$ is smooth. We denote by $f_!\colon Sm_S{}' \to Sm_S$ the composition by f. It is left adjoint to $f^{-1}\colon Sm_S \to Sm_S{}'$. For any sheaf F on Sm_S, $F \circ f_!$ is a sheaf on $Sm_S{}'$. The functor

$$Shv(Sm_S) \to Shv(Sm_S{}'),\ F \mapsto F \circ f_!$$

clearly commutes to colimits and its value on the sheaf X, with $X \in Sm_S$, is the sheaf represented by $f^{-1}(X)$. Thus this functor $Shv(Sm_S) \to Shv(Sm_S{}')$, $F \mapsto F \circ f_!$ is canonically isomorphic to

$$f^* =\colon Shv(Sm_S) \to Shv(Sm_S{}')$$

This new "explicit formula" for f^* shows that in that case f^* commutes to all limits, and thus admits a left adjoint

$$f_{\#}: Shv(Sm_{S'}) \to Shv(Sm_S)$$

Moreover, one easily checks the property that $f_{\#}(X') = f_!(X')$ and deduces the "projection" formula for any $F \in Shv(Sm_{S'})$ and any $X \in Sm_S$

$$f_{\#}(F \times f^{-1}(X)) = f_{\#}(F) \times X$$

which follows, as in [31, Proposition 1.23, p. 104], from the case of the representable sheaves.

LEMMA 5.1.1.
(1) *The functor* $f^*: Sp^{S^1}(Sm_S) \to Sp^{S^1}(Sm_{S'})$ *admits a left adjoint denoted by*

$$f_{\#}: Sp^{S^1}(Sm_{S'}) \to Sp^{S^1}(Sm_S)$$

which satisfies the following projection formula for any $F \in Sp^{S^1}(Sm_{S'})$ *and* $X \in Sm_S$

$$f_{\#}(F \wedge (f^{-1}(X)_+)) = f_{\#}(F) \wedge (X_+)$$

(2) *For any* $E \in Sp^{S^1}(Sm_S)$, *and any* $X \in Sm_S$, *the obvious morphism of* S^1-*spectra*

$$f^*(\underline{Hom}_\bullet(X_+, E)) \to \underline{Hom}_\bullet(f^{-1}(X)_+, f^*(E))$$

is an isomorphism.

Proof. (1) Follows at once from the projection formula for sheaves of sets above. (2) is an immediate consequence, by adjunction. □

LEMMA 5.1.2. *Let* $f: S' \to S$ *be an* S-*scheme which is a filtering limit of a diagram* $\{S_\alpha\}_\alpha$ *of smooth* S-*schemes with affine transition morphisms* [14, 8.2]. *For each* α *denote by* $f_\alpha: S_\alpha \to S$ *the smooth structural morphisms. Then*

(1) *For any* $X \in Sm_S$ *and* $F \in Shv(Sm_S)$ *the map*

$$colim_\alpha Hom_{Shv(Sm_{S_\alpha})}(f_\alpha^*(X), f_\alpha^*(F)) \to Hom_{Shv(Sm_{S'})}(f^*(X), f^*(F))$$

is a bijection. In particular, the functor $f^*: Shv(Sm_S) \to Shv(Sm_{S'})$ *is exact.*

(2) *For any pointed* $X \in Sm_S$ *and any* $E \in Sp^{S^1}(Sm_S)$, *the morphism of* S^1-*spectra in* $Sm_{S'}$

$$f^*(\underline{Hom}_\bullet(X, E)) \to \underline{Hom}_\bullet(f^{-1}(X), f^*E)$$

is an isomorphism.

Proof. We freely use the results in [14, 8.2]. 1) We know that there is a set I and an equalizer sequence (in $Shv(Sm_S)$) of the form

$$\amalg_{(i,j)\in I^2} Z_{i,j} \rightrightarrows \amalg_{i\in I} Y_i \to F$$

with the $Z_{i,j}$ and the Y_i in Sm_S. As f^* is a left adjoint we still have the exact sequence (in $Shv(Sm_S{}'))$

$$\amalg_{(i,j)\in I^2} f^* Z_{i,j} \rightrightarrows \amalg_i f^* Y_i \to f^* F$$

Given $\phi \in Hom_{Shv(Sm_{S'})}(f^*X, f^*(F))$, there exists a Nisnevich covering $\{X_\ell \to X\}$ (finitely many ℓ's) and morphisms $\phi_\ell \colon X_\ell \to Y_{i_\ell}$ lifting ϕ. Moreover one can find for each ℓ a Nisnevich covering $\{X'_{\ell,\ell'\mu} \to X_\ell \times_X X_{\ell'}\}$ and morphisms $X'_{\ell,\ell',\mu} \to Z_{i_\ell,i_{\ell'}}$ satisfying an obvious descent condition.

By [14, 8.2], there exists an α such that each $\phi_\ell \colon X_\ell \to Y_{i_\ell}$ is induced by some $\phi_{\ell,\alpha} \colon X_{\ell,\alpha} \to Y_{\ell,\alpha}$ and each morphism: $X'_{\ell,\ell',\mu} \to Z_{i_\ell,i_{\ell'}}$ is induced by some: $X'_{\ell,\ell',\mu,\alpha} \to Z_{i_\ell,i_{\ell'},\alpha}$ (with obvious notations).

Moreover enlarging α if necessary, we may assume that $\{X_{\ell,\alpha} \to X_\alpha\}$ is a Nisnevich covering, $\{X'_{\ell,\ell',\mu,\alpha} \to X_{\ell,\alpha} \times_{X_\alpha} X_{\ell',\alpha}\}$ is a Nisnevich covering, and that the obvious descent condition is satisfied. As the sequence of sheaves (on S_α)

$$\amalg_{(i,j)\in I^2} f_\alpha^* Z_{i,j} \rightrightarrows \amalg_i f_\alpha^* Y_i \to f_\alpha^* F$$

is exact we get a section $\phi_\alpha \in f_\alpha^* F(f_\alpha^*(X))$ which induces ϕ. Surjectivity is proven. The Injectivity is proven in very much the same way and the details are left to the reader.

Let $x \in X \in Sm_S{}'$, and let α such that x is induced by $x_\alpha \in X_\alpha \in Sm_{S\alpha}$. Using the bijection just established, we see that given and $F \in Shv(Sm_S)$ we have a bijection (with obvious notations)

$$colim_{\beta>\alpha} f_\beta^*(F)_{x_\beta} \cong f^*(F)_x$$

which immediately implies that f^* is exact. Part (1) is proven.

Part (2) follows from (1) and some easy computations involving Lemma 5.1.1. □

EXAMPLE 5.1.3. Let $x \in X \in Sm_S$. We let S' be the spectrum of the henselization $\mathcal{O}^h_{X,x}$ of the local ring $\mathcal{O}_{X,x}$ and $f \colon S' \to S$ be the obvious morphism. Denote by $S_\alpha \to S$ the system of Nisnevich neighborhoods of x, that is to say étale morphisms $f_\alpha \colon S_\alpha \to X$ such that $f_\alpha^{-1}(x)$ has exactly one element with the same residue field as x. For any S^1-spectrum E in Sm_S then its fiber E_x at x, that is to say the S^1 spectrum (in $\mathcal{S}$) given by

$$E_x := colim_{S_\alpha \to S} E(S_\alpha)$$

can be identified, using the previous lemma, with the S^1-spectrum

$$f^*(E)(S')$$

5.2. DERIVED FUNCTORS OF BASE CHANGE

It follows from Lemma 5.1.2 that if $f: S' \to S$ satisfies the assumptions of the Lemma the functor f^* is exact, and thus preserves (stable) weak equivalences. In fact we have an identification for $E \in Sp^{S^1}(Sm_S)$, $n \in \mathbb{Z}$,

$$\pi_n(f^*E) = f^*(\pi_n(E))$$

In general, however, neither $f_*: Sp^{S^1}(Sm_S') \to Sp^{S^1}(Sm_S)$ nor $f^*: Sp^{S^1}(Sm_S) \to Sp^{S^1}(Sm_S')$ do preserve stable weak equivalences (nor stable $\mathbb{A}^1$-weak equivalences); see [31, p. 62] for a precise account in the case of simplicial sheaves.

For any S^1-spectrum F in $Sm_{S'}$, we set $Rf_*(F) := f_*(F_f)$ (where as usual $(-)_f$ means a chosen fibrant resolution functor on $Sp^{S^1}(Sm_S'))$.

LEMMA 5.2.1.

(1) *The functor* $Rf_*, Sp^{S^1}(Sm_S') \to Sp^{S^1}(Sm_S)$, $E \mapsto Rf_*(E)$ *maps stable weak equivalences to stable weak equivalences. We still denote by*

$$Rf_*: \mathcal{SH}_s^{S^1}(Sm_S') \to \mathcal{SH}_s^{S^1}(Sm_S)$$

the induced functor. It is the right derived functor of f_* *in the sense of Quillen [34, I.4].*

(2) *For any* $U \in Sm_S$, *any* $n \in \mathbb{Z}$ *the canonical map*

$$\left[(f^{-1}(U)_+)[n], E\right]_{\mathcal{SH}_s^{S^1}(Sm_S')} \to [(U_+)[n], Rf_*(E)]_{\mathcal{SH}_s^{S^1}(Sm_S)}$$

is an isomorphism.

Proof. Let $U \in Sm_S$, $n \geqslant 0$. Then

$$\pi((f^{-1}(U)_+)[n], E_f)_{\mathcal{SH}_s^{S^1}(Sm_S')} = [(f^{-1}(U)_+)[n], E]_{\mathcal{SH}_s^{S^1}(Sm_S')}$$

because E_f is fibrant. But observe that the S^1-spectrum $f_*(E_f)$ is a B.G.-S^1-spectrum as it follows at once by adjunction and because $f^*(U)$ is the sheaf represented by $f^{-1}(U)$. Thus $\pi((U_+)[n], f_*(E_f)) = [(U_+)[n], Rf_*(E)]$, as well. To prove (2) in the case $n \geqslant 0$ we just observe that the adjunction induces a bijection

$$\pi((U_+)[n], f_*(E_f)) = \pi((f^{-1}(U)_+)[n], E_f)$$

But the case $n < 0$ follows easily by shifting E. Now (1) follows from (2) because the (U_+)'s generate the triangulated category $\mathcal{SH}_s^{S^1}(Sm_S)$. □

COROLLARY 5.2.2. *With the assumptions and notations above:*

(1) *given any family $\{F_\alpha\}_\alpha$ of S^1-spectra on $Sm_S{}'$, the morphism*

$$\vee_\alpha Rf_*(F_\alpha) \to Rf_*(\vee_\alpha F_\alpha)$$

is an isomorphism.

(2) *The functor Rf_* admits a left adjoint, denoted by Lf^*, which is the left derived functor of f^* in the sense of Quillen* [34, I.4]. *For any $U \in Sm_S$, the obvious morphism*

$$(f^{-1}(U)_+) \to Lf^*(U_+)$$

is an isomorphism.

Proof. To show (1), it is sufficient to check that it induces an isomorphism when applying $[(U_+)[n], -]$ for any $U \in Sm_S$, any $n \in \mathbb{Z}$. But this follows from (2) of Lemma 5.2.1 by 3.1.1 (4).

The existence of the left adjoint in (2) is equivalent to proving that for each S^1-spectra E on Sm_S, the functor $F \mapsto [E, Rf_*(F)]_{\mathcal{SH}_s^{S^1}(Sm_S)}$ is representable.[13] That statement is true for $E = (U_+)$ by Lemma 5.2.1 (2). But clearly the conclusion is stable under taking cones, arbitrary wedges, and we conclude because the smallest subcategory of $\mathcal{SH}_s^{S^1}(Sm_S)$ satisfying these properties and containing the (U_+)'s is clearly $\mathcal{SH}_s^{S^1}(Sm_S)$ itself. □

LEMMA 5.2.3.

(1) *Let $F \in Sp^{S^1}(Sm_S{}')$ be an $\mathbb{A}^1$-local spectrum. Then $Rf_*(F)$ is $\mathbb{A}^1$-local.*

(2) *Let $F \in Sp^{S^1}(Sm_S{}')$ be an S^1-spectrum. Then the canonical morphism*

$$L_{\mathbb{A}^1}(Rf_*(F)) \to Rf_*(L_{\mathbb{A}^1}(F))$$

is an isomorphism. As a consequence, Rf_* does preserve stable $\mathbb{A}^1$-weak equivalences and induces a functor denoted

$$R^{\mathbb{A}^1} f_*: \mathcal{SH}_{\mathbb{A}^1}^{S^1}(Sm_S{}') \to \mathcal{SH}_{\mathbb{A}^1}^{S^1}(Sm_S)$$

[13] Of the form $F \mapsto [f^*(E), F]_{\mathcal{SH}_s^{S^1}(Sm_S{}')}$.

(3) *The functor* $Lf^*: \mathcal{SH}_s^{S^1}(Sm_S) \to \mathcal{SH}_s^{S^1}(Sm_{S'})$ *does preserve stable* $\mathbb{A}^1$-*weak equivalences and induces a functor denoted*

$$L_{\mathbb{A}^1} f^*: \mathcal{SH}_{\mathbb{A}^1}^{S^1}(Sm_S) \to \mathcal{SH}_{\mathbb{A}^1}^{S^1}(Sm_{S'})$$

which is left adjoint to $R^{\mathbb{A}^1} f_*$.

Proof. (1) clearly follows from Lemma 5.2.1 (1). (2) follows from (1) and the description of the localization functor $L_{\mathbb{A}^1}$ we gave above, taking into account the formula $Rf_*(F^{\mathbb{A}^1}) = (Rf_*(F))^{\mathbb{A}^1}$ (which is a consequence of Lemma 5.2.1 (1)), and the fact that Rf_* commutes to direct sums (and thus to forming telescopes). (3) follows from (1) by adjunction. □

We also mention the following:

LEMMA 5.2.4. *For any morphism* $f: S' \to S$ *and any integer* $n \in \mathbb{Z}$, *the functor*

$$Lf^*: \mathcal{SH}_s^{S^1}(Sm_S) \to \mathcal{SH}_s^{S^1}(Sm_{S'})$$

maps n-*connected* S^1-*spectra to* n-*connected* S^1-*spectra.*

Proof. Indeed it is sufficient to treat the case $n = -1$. *This follows clearly from Lemma 3.3.4 and the fact that for* $X \in Sm_S$, $Lf^*(X_+) = (f^{-1}(X)_+)$. □

Remark 5.2.5. In case f is a finite morphism, we can prove as in [31, Propositions 1.27 & 2.12] that $Rf_* = f_*$. Thus in that case as well, the exact functor $Rf_* = f_*$ maps n-connected S^1-spectra to n-connected S^1-spectra.

Functoriality with respect to smooth morphisms. Assume now that $f: S' \to S$ is a smooth morphism. Then we already mentioned that f^* preserves stable weak equivalences and thus it is clear that

$$f^* = Lf^*: \mathcal{SH}_s^{S^1}(Sm_S) \to \mathcal{SH}_s^{S^1}(Sm_{S'})$$

LEMMA 5.2.6. *With the previous assumptions and notations:*

(1) *the functor* $Lf^* = f^*$ *admits a left adjoint, denoted by* $Lf_\#$, *which is the left derived functor of* $f_\#$ *in the sense of Quillen* [34, I.4]. *For any* $U \in Sm_{S'}$, *the obvious morphism*

$$(f_\#(U)_+) \to Lf_\#((U_+))$$

is an isomorphism.

(2) *If* $E \in Sp^{S^1}(Sm_S)$ *be a B.G.-spectrum (resp.* $\mathbb{A}^1$-*local spectrum), then so is* $f^*(E)$.

(3) *Let* $E \in Sp^{S^1}(Sm_S)$ *be a fibrant spectrum. Then the canonical morphism*

$$L_{\mathbb{A}^1}(f^*(F)) \to f^*(L_{\mathbb{A}^1}(F))$$

is an isomorphism.

(4) *The functor* $Lf_\#: \mathcal{SH}_s^{S^1}(Sm_S{}') \to \mathcal{SH}_s^{S^1}(Sm_S)$ *preserves stable* $\mathbb{A}^1$*-weak equivalences and the induced functor*

$$Lf_\#^{\mathbb{A}^1}: \mathcal{SH}_{\mathbb{A}^1}^{S^1}(Sm_S{}') \to \mathcal{SH}_{\mathbb{A}^1}^{S^1}(Sm_S)$$

is left adjoint to $L_{\mathbb{A}^1} f^*$.

Proof. (1) The functor f^* being a left adjoint, it commutes to arbitrary sums and thus we formally get as in Corollary 5.2.2 the existence of a left adjoint $Lf_\#^{\mathbb{A}^1}$. To prove the last part of the statement we first observe that if $E \in Sp^{S^1}(Sm_S)$ is fibrant, then $f^*(E)$ is a B.G.-spectrum in $Sm_S{}'$: this follows easily from the fact that applying $f_!$ to a distinguished square over S' yields a distinguished square over S. This easily implies the statement.

(2, 3, 4) are proven in the same way as in Lemma 5.2.3. We observe here that we already knew that f^* preserves stable $\mathbb{A}^1$-weak equivalences. □

COROLLARY 5.2.7. *Let* $f: S' \to S$ *be an* S*-scheme which is a filtering limit of a diagram* $\{S_\alpha\}_\alpha$ *of smooth* S*-schemes with affine transition morphisms* [14, 8.2]. *For each* α *denote by* $f_\alpha: S_\alpha \to S$ *the smooth structural morphisms.*

(1) *for any BG-*S^1*-spectrum* E *then* $f^*(E)$ *is a BG-spectrum.*

(2) *For any* $X \in Sm_S$ *and any* $E \in Sp^{S^1}(Sm_S)$ *the morphism*

$$colim_\alpha[(f_\alpha^{-1}(X)_+), f_\alpha^* E]_{\mathcal{SH}^{S^1}(Sm_{S_\alpha})} \to [(f^{-1}(X)_+), f^* E]_{\mathcal{SH}_s^{S^1}(Sm_S{}')}$$

is an isomorphism.

(3) *For any* $E \in Sp^{S^1}(Sm_S)$ *any pointed* $X \in Sm_S$*, the canonical morphism (in* $Sp^{S^1}(Sm_S{}')$*)*

$$f^*(E^{(X)}) \to (f^*(E))^{(f^{-1}X)}$$

is an isomorphism. As a consequence, the morphism

$$f^*(L_{\mathbb{A}^1}(E)) \to L_{\mathbb{A}^1}(f^*(E))$$

is a stable weak equivalence, and in particular, if E *is* $\mathbb{A}^1$*-local so is* f^*E.

Proof. It is not hard to see, using again the results in [14, 8.2], that a distinguished square in $Sm_S{}'$ is the pull-back along one of the morphism $S' \to S_{\alpha_0}$ of a distinguished square defined on S_{α_0}. So replacing S by S_{α_0}, we may assume given a distinguished square in Sm_S. The fact that f^*E is a B.G.-spectrum follows then from: Lemma 5.1.2 (2), the fact (by Lemma 5.2.6) that each $f_\alpha^* E$ is a B.G.-spectrum, that the pull back of a distinguished square by any morphism is a distinguished square and the fact that a filtering colimit of homotopy cartesian squares of S^1-spectra is still a homotopy cartesian square of S^1-spectra. This proves (1).

Part (2) is a consequence of (1) together the corresponding isomorphism of S^1-spectra of sections

$$f^*(E)(f^{-1}(X)) = colim_\alpha f_\alpha^*(E)(f_\alpha^{-1}(X))$$

which follows from lemma 5.1.2 (1).

Point (3) follows easily from (1) and lemma 5.1.2 (2). □

EXAMPLE 5.2.8. We keep the same assumptions and notations as in 5.1.3. Then the lemma above implies that the morphisms

$$\pi_0(E(\mathcal{O}_{X,x}^h)) \to [S^0, f^*(E)]_{\mathcal{SH}_s^{S^1}(Sm_{S'})}$$

and

$$\pi_0(L_{\mathbb{A}^1}(E)(\mathcal{O}_{X,x}^h)) \to [S^0, f^*(E)]_{\mathcal{SH}_{\mathbb{A}^1}^{S^1}(Sm_{S'})}$$

are isomorphisms. As a consequence the S^1-spectrum $L_{\mathbb{A}^1}(E)_x$, fiber at x of $L_{\mathbb{A}^1}(E)$, is isomorphic in the "usual" stable homotopy category of S^1-spectra to $(L_{\mathbb{A}^1}(f^*E))(Spec\mathcal{O}_{X,x}^h)$.

5.3. THE GLUING THEOREM

Let $j: U \to S$ be an open immersion. We let Sm_{SU} denote the category of smooth U-schemes. We let $Shv(Sm_U)$ denote the corresponding category of sheaves of sets in the Nisnevich topology. The restriction functor $j^*: Shv(Sm_S) \to Shv(Sm_U)$ has both a right adjoint j_* and a left adjoint $j_\#$ and is thus exact. The functor $j_\#: Shv(Sm_U) \to Shv(Sm_S)$ is a fully faithful embedding which identifies $Shv(Sm_U)$ with the category $Shv(Sm_S)/U$ of objects $F \in Shv(Sm_S)$ whose structure morphism $F \to *$ factors through the sub-object $*_U = U \subset S = * \in Sm_S$ of $*$.

LEMMA 5.3.1.

(1) *The functor*

$$j_\#: Shv(Sm_U) \to Shv(Sm_S)$$

is exact and in particular the induced functor on spectra

$$j_\#: Sp^{S^1}(Sm_U) \to Sp^{S^1}(Sm_S)$$

preserves stable weak equivalences and maps n-connected S^1-spectra to n-connected S^1-spectra.

(2) *The induced functor*

$$j_\# = Lj_\#: \mathcal{SH}_s^{S^1}(Sm_{SU}) \to \mathcal{SH}_s^{S^1}(Sm_S)$$

is a fully faithful embedding.

Proof. (1) is proven by computing the fibers for instance.

(2) Indeed, the natural transformation $Id \to Lj^* \circ Lj_\# = j^* \circ j_\#$ is an isomorphism. Thus $[E,F]_{\mathcal{SH}_s^{S^1}(Sm_{SU})} = [E, Lj^* \circ Lj_\#(F)] = [Lj_\#(E), Lj_\#(F)]$, thus proving the first assertion. □

Let now $i: Z \to S$ be the complementary closed immersion of the open immersion $j: U \to S$ (with the reduced induced structure on Z). We let $Sm_{SZ} := \mathbf{Sm}_Z$ be the category of smooth Z-schemes and let $Shv(Sm_Z)$ denote the corresponding category of sheaves in the Nisnevich topology. The functor i_* which is exact [31, Prop. 1.27 p. 105]; as in [31, Prop. 2.12 p. 108] we deduce that i_* preserves A^1-weak equivalences and $\mathbb{A}^1$-local objects and thus in particular it also preserves the $\mathbb{A}^1$-localization.

Given an S^1-spectrum $E \in Sp^{S^1}(Sm_S)$, the composition

$$j_\# E_U \to E \to i_* i^*(E)$$

is the trivial morphism in $Sp^{S^1}(Sm_S)$, so that we have a canonical morphism of S^1-spectra $C(j_\# j^*(E) \to E) \to i_* i^*(E)$. For simplicity in the sequel we will simply set $E_U := j^* E$.

The following lemma can be either deduced or directly proven using [31, Theorem 2.21]:

LEMMA 5.3.2. *For any S^1-spectrum E on Sm_S, the morphism*

$$C(j_\# E_U \to E) \to i_* L_{\mathbb{A}^1}(Li^*(E))$$

is an $\mathbb{A}^1$-weak equivalence. Thus there is a canonical exact triangle in $\mathcal{SH}_s^{S^1}(Sm_S)$:

$$L_{\mathbb{A}^1}\big(j_\#(E_U)\big) \to L_{\mathbb{A}^1}(E) \to i_*(L_{\mathbb{A}^1}(Li^*(E))$$

Remark 5.3.3. As J. Ayoub pointed out to us, contrary to what we previously thought, in general if E is an $\mathbb{A}^1$-local S^1-spectrum on Sm_{SU} then $j_\#(E)$ is not $\mathbb{A}^1$-local. For instance take E to be the Eilenberg–MacLane spectrum $H(\mathbb{G}_m)|U$ over U. Then using the previous lemma we see that the $\mathbb{A}^1$-localization of $j_\# E$ is the fiber of

$$H\mathbb{G}_m \to i_* H\mathbb{G}_m$$

because the two spectra are clearly $\mathbb{A}^1$-local. It is easy to check it is not simplicially equivalent to $j_\#(H(\mathbb{G}_m)|U)$.

6. The $\mathbb{A}^1$-Connectivity Theorems

6.1. THE $\mathbb{A}^1$-CONNECTIVITY THEOREM OVER A FIELD

In this section, S is assumed to be the spectrum of the field k.

DEFINITION 6.1.1. (1) Let $\mathcal{X}$ be an $\mathbb{A}^1$-local pointed simplicial sheaf on Sm_S and let $n \in \mathbb{N}$ be an integer. We say that $\mathcal{X}$ is *weakly n-connected* if and only if for any generic point $\eta \in X \in Sm_S$ with residue field F the fiber $\mathcal{X}_\eta = \mathcal{X}(F)$ is n-connected (compare [27, Definition 3.3.5]).

(2) Let E be an S^1-spectrum on Sm_S and let $n \in \mathbb{Z}$ be an integer. We say that E is *weakly n-connected* if and only if for any generic point $\eta \in X \in Sm_S$ with residue field F the fiber $E_\eta = E(F)$ is an n-connected S^1-spectrum.

Remark 6.1.2. Clearly any (-1)-connected S^1-spectrum on Sm_S is weakly (-1)-connected. If one assumes E is fibrant and $\mathbb{A}^1$-local Remark 4.1.3 implies that each E_n is a fibrant $\mathbb{A}^1$-local pointed simplicial sheaf. Moreover, the assumption that E is weakly (-1)-connected is equivalent to requiring that E_n is weakly $(n-1)$-connected for each integer $n \in \mathbb{N}$.

We will now prove that [27, Lemma 3.3.6] holds in a more general context:

LEMMA 6.1.3. *Let $\mathcal{X}$ be an $\mathbb{A}^1$-local pointed simplicial sheaf on Sm_S and $n \geqslant 0$ an integer. Then the following conditions are equivalent:*

(i) $\mathcal{X}$ is weakly n-connected;
(ii) $\mathcal{X}$ is n-connected.

The same proof carries over exactly the same way so that we end up with proving the following generalization of [27, Lemma 3.3.7]:

LEMMA 6.1.4. *Let $X \in Sm_S$ be irreducible and let $\Omega \in X$ be an open dense subscheme. Then the pointed simplicial sheaf*

$$L_{\mathbb{A}^1}(X/\Omega)$$

is 0-connected.

Proof. [27, Lemma 3.3.7] gives exactly the statement when k is perfect (the proof relies on the homotopy purity theorem [31] and the perfectness of the base field). Thus we may assume that k is not perfect!

Recall that by [31, Corollary 3.22 p. 94], for any simplicial sheaf of sets $\mathcal{X}$ the morphism of sheaves

$$\mathcal{X}_0 \to \pi_0(L_{\mathbb{A}^1}(\mathcal{X}))$$

is an epimorphism.

In particular,

$$X \to \pi_0(L_{\mathbb{A}^1}(X/\Omega))$$

is an epimorphism. Thus it is sufficient to show that any point $x \in X$ admits an open neighborhood U such that $\pi_0(L_{\mathbb{A}^1}(U/(\Omega \cap U))$ is trivial; indeed by functoriality then, $U \to \pi_0(L_{\mathbb{A}^1}(X/\Omega))$ is trivial.

Let $Z \subset X$ denote the closed immersion of the complement of Ω (with the reduced structure). As we assumed k is not perfect, it is thus infinite and by Gabber's presentation Lemma [13, Lemma 3.1] or more precisely [9, Theorem 3.1.1] any point $x \in X$ admits an open neighborhood U and an étale morphism $\pi: U \to \mathbb{A}^1_V$, with V some open subscheme in the affine space $\mathbb{A}^{d-1}$, with d the dimension of X at x, such that π induces a closed immersion $Z_U := Z \cap U \to \mathbb{A}^1_V$, satisfying $Z_U = \pi^{-1}(\pi(Z_U))$ and such that $Z_U \to V$ is finite. One thus gets an isomorphism of Nisnevich sheaves $U/(U - Z_U) \to \mathbb{A}^1_V/(\mathbb{A}^1_V - Z_U)$ and it suffices to check that $\pi_0(L_{\mathbb{A}^1}(\mathbb{A}^1_V/(\mathbb{A}^1_V - Z_U)))$ is trivial.

We now follow [9]: because $Z_U \to V$ is finite, $Z_U \to \mathbb{P}^1_V$ is proper, thus still a closed immersion, and moreover it doesn't meet the section at infinity $s_\infty: V \to \mathbb{P}^1_V$. But now by Mayer–Vietoris excision, the morphism of sheaves

$$\mathbb{A}^1_V/(\mathbb{A}^1_V - Z_U) \to \mathbb{P}^1_V/(\mathbb{P}^1_V - Z_U)$$

is an isomorphism. But as $\mathbb{A}^1_V \to \mathbb{P}^1_V/(\mathbb{P}^1_V - Z_U)$ is onto and $L_{\mathbb{A}^1}(\mathbb{A}^1_V) = L_{\mathbb{A}^1}(V)$ is an isomorphism, the composition

$$V \to \mathbb{A}^1_V \to \pi_0(L_{\mathbb{A}^1}(\mathbb{P}^1_V/(\mathbb{P}^1_V - Z_U)))$$

is onto for any section $V \to \mathbb{A}^1_V$, for instance the zero section.

But in $\mathbb{P}^1_V$ the zero section is $\mathbb{A}^1$-homotopic to the section at infinity $s_\infty: V \to \mathbb{P}^1$. As $s_\infty(V) \subset \mathbb{P}^1_V - Z_U$ we see that

$$V \to \pi_0(L_{\mathbb{A}^1}(\mathbb{P}^1_V/(\mathbb{P}^1_V - Z_U)))$$

is the trivial morphism, as required. □

Remark 6.1.5. The analogue of the previous Lemma is wrong in general over a base S which is not a field. The smallest counter-example is obtained as follows: take $X = S$ and $i: Z \subset S$ a non-empty closed subscheme of codimension $d > 0$. Then using Lemma 5.3.2 and the fact that $i_*(L_{\mathbb{A}^1}(Li^*S^0)) = i_*(S^0)$ one gets that $L_{\mathbb{A}^1}(S/S - Z) = i_*(S^0)$, which is clearly not 0-connected.

Now we can easily prove:

LEMMA 6.1.6. *Let E be an $\mathbb{A}^1$-local S^1-spectrum. Then the following conditions are equivalent:*

(i) E is weakly (-1)-connected;
(ii) E is (-1)-connected.

Proof. We may assume E to be fibrant and $\mathbb{A}^1$-local; Remark 4.1.3 implies that each E_n is a fibrant $\mathbb{A}^1$-local simplicial sheaf. Moreover, by Remark 6.1.2 each E_n is weakly $(n-1)$-connected. Then Lemma 6.1.3 gives us that each E_n is indeed $(n-1)$-connected, so that E is (-1)-connected. □

LEMMA 6.1.7. *Let E be a (-1)-connected S^1-spectrum. Then its $\mathbb{A}^1$-localization is weakly (-1)-connected.*

Proof. Let $\eta \in X \in Sm_S$ be a generic point of X with residue field F. By Corollary 5.2.7, the fiber $L_{\mathbb{A}^1}(E)_\eta$, can be identified (up to isomorphism in $\mathcal{SH}$) to the S^1-spectrum of simplicial sets

$$L_{\mathbb{A}^1}(f^*(E))(Spec(F))$$

where $f\colon Spec(F) \to Spec(k)$ is the obvious morphism. But clearly $f^*(E)$ is still (-1)-connected. Then by Corollary 4.3.3, the groups

$$[(Spec(F)_+)[n], L_{\mathbb{A}^1}(f^*(E))] = \pi_n(L_{\mathbb{A}^1}(f^*(E))(Spec(F)))$$

vanish for $n < 0$. Thus $L_{\mathbb{A}^1}(f^*(E))(Spec(F)) \cong L_{\mathbb{A}^1}(E)_\eta$ is (-1)-connected, proving that $L_{\mathbb{A}^1}(E)$ is weakly (-1)-connected. □

The Lemmas above imply our main result:

THEOREM 6.1.8. *Let E be an (-1)-connected S^1-spectrum on Sm_S. Then its $\mathbb{A}^1$-localization $L_{\mathbb{A}^1}(E)$ is (-1)-connected.*

6.2. $\mathbb{A}^1$-CONNECTIVITY AND STRICTLY $\mathbb{A}^1$-INVARIANT SHEAVES

In this section now, S is again a general base scheme.

DEFINITION 6.2.1. A sheaf of abelian groups $M \in \mathbb{A}b(Sm_S)$ is said to be *strictly $\mathbb{A}^1$-invariant* if and only if for any $X \in Sm_S$ and any integer $n \in \mathbb{N}$ the obvious homomorphism

$$H^n_{Nis}(X; M) \to H^n_{Nis}(X \times \mathbb{A}^1; M)$$

is an isomorphism. We denote by $\mathbb{A}b_{\mathbb{A}^1}(Sm_S) \subset \mathbb{A}b(Sm_S)$ the full subcategory consisting of abelian sheaves which are strictly $\mathbb{A}^1$-invariant.

The following justifies the introduction of the previous notion:

LEMMA 6.2.2. *Let $M \in \mathbb{A}b(Sm_S)$. Then the Eilenberg–MacLane S^1-spectrum $H(M)$ is $\mathbb{A}^1$-local if and only if M is strictly $\mathbb{A}^1$-invariant.*

Proof. Assume $H(M)$ is $\mathbb{A}^1$-local. Lemma 4.1.2 (iv) and the identification $H^n_{Nis}(X; M) = [(X_+), HM[n]]$ imply that for any $X \in Sm_S$ and $n \in \mathbb{Z}$, M is strictly $\mathbb{A}^1$-invariant.

Conversely, if M is strictly $\mathbb{A}^1$-invariant, then we deduce that the homomorphism

$$[(X_+), HM[n]] \to [(X_+) \wedge (\mathbb{A}^1_+), HM[n]]$$

is an isomorphism for all $X \in Sm_S$ and all $n \in \mathbb{Z}$, and we conclude by Lemma 4.1.2 that HM is $\mathbb{A}^1$-local. □

EXAMPLE 6.2.3. Let M be an abelian group; then its associated constant sheaf M on $Sm_{S_{Nis}}$ is strictly $\mathbb{A}^1$-invariant: we mentioned 3.2.4 that $H^*(X; M)$ vanishes for $* > 0$ and clearly $X \mapsto M(X)$ is a $\mathbb{A}^1$-invariant sheaf.

EXAMPLE 6.2.4. Let M be a homotopy invariant sheaf with transfers on the category Sm_k of smooth k-schemes in the sense of Voevodsky [40]. When k is perfect, one of the main results of [39] is that M is strictly $\mathbb{A}^1$-invariant.

EXAMPLE 6.2.5. Let $f : S' \to S$ be a smooth morphism (or a morphism satisfying the assumption of Corollary 5.2.7). Then for any strictly $\mathbb{A}^1$-invariant sheaf M on S, the sheaf $f^*(M)$ on S' is strictly $\mathbb{A}^1$-invariant. This follows for instance from Corollary 5.2.7 and the fact that $f^*(HM) = Hf^*(M)$.

Consequence of the stable $\mathbb{A}^1$-connectivity property.

LEMMA 6.2.6. *Assume stable $\mathbb{A}^1$-connectivity propertyholds over S. Let E be an $\mathbb{A}^1$-local S^1-spectrum. Then its non-negative part $E_{\geqslant 0}$ is an $\mathbb{A}^1$-local S^1-spectrum. As a consequence, for any integer $n \in \mathbb{Z}$ the triangle of S^1-spectra (in $\mathcal{SH}^{S^1}_s(Sm_S)$)*

$$E_{\geqslant n} \to E \to E_{\leqslant n-1}$$

consists of $\mathbb{A}^1$-local S^1-spectra.

Proof. Indeed, applying the $\mathbb{A}^1$-localization functor yields

$$L_{\mathbb{A}^1}(E_{\geqslant 0}) \to L_{\mathbb{A}^1}(E) \cong E$$

By assumption, $L_{\mathbb{A}^1}(E_{\geqslant 0})$ is (-1)-connected. Thus by the property of the standard t-structure (see Section 3), $L_{\mathbb{A}^1}(E_{\geqslant 0})$ maps back to $E_{\geqslant 0}$ and $E_{\geqslant 0}$, being a direct factor of $L_{\mathbb{A}^1}(E_{\geqslant 0})$, is $\mathbb{A}^1$-local. The rest of the statement easily follows. □

THEOREM 6.2.7. *Assume stable $\mathbb{A}^1$-connectivity propertyholds on S. Let E be an S^1-spectrum over S, then the following properties are equivalent*

(i) E is $\mathbb{A}^1$-local;
(ii) Each of the S^1-spectra $E_{\geqslant n}$, $n \in \mathbb{Z}$, is $\mathbb{A}^1$-local;
(iii) the sheaves $\pi_n(E) \in \mathbb{A}b(Sm_S)$ are strictly $\mathbb{A}^1$-invariant for each $n \in \mathbb{Z}$.

Proof. We already know from Lemma 6.2.6 that (i)⇒(ii). Clearly, because of the triangles

$$H(\pi_n(E))[n] \to E_{\geqslant n} \to E_{\geqslant n-1}$$

we get (ii)⇒(iii). The implication (iii)⇒(i) easily follows from Lemmas 3.3.3, 4.1.2, 6.2.2. □

Remark 6.2.8. If $E \in Sp$ is an S^1-spectrum in $\mathcal{S}$, we see from the previous Theorem and Example 6.2.3 that E is $\mathbb{A}^1$-local (when considered in $Sp^{S^1}(Sm_S)$). We thus deduce from Remark 3.3.5 that the functor

$$\mathcal{SH} \to \mathcal{SH}^{S^1}_{\mathbb{A}^1}(Sm_S),\, E \mapsto E$$

is a fully faithful embedding when S is irreducible. This holds without any assumption on S. In particular, the canonical morphism

$$\mathbb{Z} \to [S^0, S^0]_{\mathbb{A}^1}$$

is an isomorphism.

We can now prove the Lemma 5 of the introduction which we restate:

COROLLARY 6.2.9. *Assume that the stable $\mathbb{A}^1$-connectivity property holds on S. Then:*

(1) *For any sheaf E of S^1-spectra over S and any integer $n \in \mathbb{Z}$, the sheaves*

$$\pi_n^{\mathbb{A}^1}(E) := \pi_n(L_{\mathbb{A}^1}(E))$$

are strictly $\mathbb{A}^1$-invariant.

(2) *Let* $E\colon (Sm_S)^{op} \to$, $U \mapsto E(U)$ *be a presheaf of* S^1*-spectra over* S *which has the B.G. property for distinguished squares (see* 3.1.6 *and* 3.1.8, *or* [31]*) and the* $\mathbb{A}^1$*-invariance property: for any* $U \in Sm_S$, *the morphism* $E(U) \to E(U \times \mathbb{A}^1)$ *is a stable weak equivalence. Then for any* $n \in \mathbb{Z}$, *the Nisnevich sheaf* $\mathcal{H}^n(E) = \pi_{-n}(E)$ *associated to the presheaf* $U \mapsto E^n(U) := \pi_{-n}(E(U))$ *is a strictly* $\mathbb{A}^1$*-invariant sheaf.*

Proof. We already mentioned that for an S^1-spectrum E the sheaf $\pi_{-n}^{\mathbb{A}^1}(E)$ is the associated Nisnevich sheaf to the presheaf $U \mapsto [(U_+), E[n]]_{\mathbb{A}^1}$; according to Theorem 6.2.7 the statement becomes clear (the case of presheaves uses Remark 4.2.6). □

The homotopy t-structure.

DEFINITION 6.2.10. (1) For any S^1-spectrum E and any integer $n \in \mathbb{Z}$ we set

$$\pi_n^{\mathbb{A}^1}(E) := \pi_n L_{\mathbb{A}^1}(E) \in \mathbb{A}b_{\mathbb{A}^1}(Sm_S)$$

(2) An S^1-spectrum F is said to be $\mathbb{A}^1$-non-positive if $L_{\mathbb{A}^1}(F) \in \mathcal{SH}_s^{S^1}(Sm_S)_{\leqslant 0}$ that is to say if

$$\pi_n^{\mathbb{A}^1}(F) = 0 \quad \text{for } n > 0$$

We denote by $\mathcal{SH}_{\mathbb{A}^1}^{S^1}(Sm_S)_{\leqslant 0} \subset \mathcal{SH}_{\mathbb{A}^1}^{S^1}(Sm_S)$ the full subcategory whose objects are $\mathbb{A}^1$-non-positive.

(3) We say that an S^1-spectrum E is $\mathbb{A}^1$-non-negative if $L_{\mathbb{A}^1}(E) \in \mathcal{SH}_s^{S^1}(Sm_S)_{\geqslant 0}$ that is to say if

$$\pi_n^{\mathbb{A}^1}(E) = 0 \quad \text{for } n < 0$$

We denote $\mathcal{SH}_{\mathbb{A}^1}^{S^1}(Sm_S)_{\geqslant 0} \subset \mathcal{SH}_{\mathbb{A}^1}^{S^1}(Sm_S)$ the full subcategory whose objects are $\mathbb{A}^1$-non-negative.

The results of Section 3 and the previous ones clearly imply the following Lemma, in which we set for $E \in Sp^{S^1}(Sm_S)$ and $n \in \mathbb{Z}$, $E_{\geqslant n}^{\mathbb{A}^1} := (L_{\mathbb{A}^1}(E))_{\geqslant n}$ and $E_{\leqslant n}^{\mathbb{A}^1} := (L_{\mathbb{A}^1}(E))_{\leqslant n}$:

LEMMA 6.2.11. *Assume stable* $\mathbb{A}^1$*-connectivity propertyholds over* S. *The pair* $(\mathcal{SH}_{\mathbb{A}^1}^{S^1}(Sm_S)_{\geqslant 0}, \mathcal{SH}_{\mathbb{A}^1}^{S^1}(Sm_S)_{\leqslant 0})$ *defines a t-structure [5] on* $\mathcal{SH}_{\mathbb{A}^1}^{S^1}(Sm_S)$. *The functor*

$$L_{\mathbb{A}^1}\colon \mathcal{SH}_{\mathbb{A}^1}^{S^1}(Sm_S) \to \mathcal{SH}_s^{S^1}(Sm_S)$$

is exact with respect to the t-structures (the standard one on the right). For any $E \in Sp^{S^1}(Sm_S)$*, the morphisms:*

$$hocolim_{n\to-\infty} E^{\mathbb{A}^1}_{\geqslant n} \to E \text{ and } E \to holim_{n\to+\infty} E^{\mathbb{A}^1}_{\leqslant n}$$

are both isomorphisms in $\mathcal{SH}^{S^1}_{\mathbb{A}^1}(Sm_S)$.

For any $U \in Sm_S$ *of Krull dimension* $\leqslant d$*, any* $E \in Sp^{S^1}(Sm_S)$ *the morphism*

$$[(U_+), E]_{\mathbb{A}^1} \to [(U_+), E_{\leqslant n}]_{\mathbb{A}^1}$$

is onto for $n \geqslant d-1$ *and an isomorphism for* $n \geqslant d$.

The following conditions on an S^1*-spectrum* F *are equivalent:*

(i) *For any integer* $n>0$*, any* $U \in Sm_S$ *the group* $[(U_+)[n], F]_{\mathbb{A}^1}$ *is trivial.*
(ii) $F \in \mathcal{SH}^{S^1}_{\mathbb{A}^1}(Sm_S)_{\leqslant 0}$.

The t-structure on $\mathcal{SH}^{S^1}_{\mathbb{A}^1}(Sm_S)$ in the Lemma will be called the *homotopy t-structure* on $\mathcal{SH}^{S^1}_{\mathbb{A}^1}(Sm_S)$, provided of course the stable $\mathbb{A}^1$-connectivity propertyholds over S.

Remark 6.2.12. In fact, without any assumption on S, one may define the subcategory $\mathcal{SH}^{S^1}_{\mathbb{A}^1}(Sm_S)_{\leqslant 0}$ by the last condition of the previous Lemma. One may then define the subcategory $\mathcal{SH}^{S^1}_{\mathbb{A}^1}(Sm_S)_{\geqslant 0}$ by required that $E \in \mathcal{SH}^{S^1}_{\mathbb{A}^1}(Sm_S)_{\geqslant 0}$ if and only if for any $F \in \mathcal{SH}^{S^1}_{\mathbb{A}^1}(Sm_S)_{\leqslant 0}$, one has $[E, F[-1]]_{\mathbb{A}^1} = 0$. It can be shown *a priori* that these classes define a t-structure on $\mathcal{SH}^{S^1}_{\mathbb{A}^1}(Sm_S)$.

Assume again now that the stable $\mathbb{A}^1$-connectivity propertyholds over S. Recall that the *heart* of the t-structure is the intersection

$$\mathcal{SH}^{S^1}_{\mathbb{A}^1}(Sm_S)_{\geqslant 0} \cap \mathcal{SH}^{S^1}_{\mathbb{A}^1}(Sm_S)_{\leqslant 0}$$

and that it is an abelian category by [5].

From Theorem 6.2.7 we see that $\pi_0^{\mathbb{A}^1}$ induces a functor

$$\pi_0^{\mathbb{A}^1}: \mathcal{SH}^{S^1}_{\mathbb{A}^1}(Sm_S) \to \mathbb{A}b_{\mathbb{A}^1}(Sm_S)$$

Moreover by Example 6.2.3, we have the functor

$$H: \mathbb{A}b_{\mathbb{A}^1}(Sm_S) \to \mathcal{SH}^{S^1}_{\mathbb{A}^1}(Sm_S)$$

whose image is clearly contained in the heart $\mathcal{SH}^{S^1}_{\mathbb{A}^1}(Sm_S)_{\geqslant 0} \cap \mathcal{SH}^{S^1}_{\mathbb{A}^1}(Sm_S)_{\leqslant 0}$. The following Lemma is then rather clear from what we have done so far:

LEMMA 6.2.13. *Assume that the stable $\mathbb{A}^1$-connectivity propertyholds over S. The functors*

$$\pi_0^{\mathbb{A}^1}: \mathcal{SH}_{\mathbb{A}^1}^{S^1}(Sm_S)_{\geqslant 0} \cap \mathcal{SH}_{\mathbb{A}^1}^{S^1}(Sm_S)_{\leqslant 0} \to \mathbb{A}b_{\mathbb{A}^1}(Sm_S)$$

and

$$H: \mathbb{A}b_{\mathbb{A}^1}(Sm_S) \to \mathcal{SH}_{\mathbb{A}^1}^{S^1}(Sm_S)_{\geqslant 0} \cap \mathcal{SH}_{\mathbb{A}^1}^{S^1}(Sm_S)_{\leqslant 0}$$

are equivalences of categories inverse to each other. As a consequence the category $\mathbb{A}b_{\mathbb{A}^1}(Sm_S)$ is abelian, and the functor $\mathbb{A}b_{\mathbb{A}^1}(Sm_S) \to \mathbb{A}b(Sm_S)$ is an exact full embedding and admits as left adjoint the functor

$$L_0^{\mathbb{A}^1}: \mathbb{A}b(Sm_S) \to \mathbb{A}b_{\mathbb{A}^1}(Sm_S), M \mapsto \pi_0^{\mathbb{A}^1}(HM)$$

Moreover, the functor

$$\begin{aligned} &(\mathbb{A}b_{\mathbb{A}^1}(Sm_S))^2 \to \mathbb{A}b_{\mathbb{A}^1}(Sm_S) \\ &(M, N) \mapsto M \otimes_{\mathbb{A}^1} N := L_0^{\mathbb{A}^1}(M \otimes N) \end{aligned}$$

defines a symmetric monoidal structure on $\mathbb{A}b_{\mathbb{A}^1}(Sm_S)$.

Remark 6.2.14. The fact that the category $\mathbb{A}b_{\mathbb{A}^1}(Sm_S)$ is abelian, and the functor $\mathbb{A}b_{\mathbb{A}^1}(Sm_S) \to \mathbb{A}b(Sm_S)$ is an exact full embedding is not trivial *a priori*. Indeed, this exactly means that if $f: M \to N$ is a morphism between strictly $\mathbb{A}^1$-invariant sheaves over S, the Kernel and Cokernel (computed in the abelian category of sheaves) are both strictly $\mathbb{A}^1$-invariant.

Remark 6.2.15. We do not know any example (see the computations in [29]) of a sheaf M where the canonical morphism

$$M \to L_0^{\mathbb{A}^1}(M)$$

is not an epimorphism in the Zariski topology.

Remark 6.2.16. On can show that the symmetric monoidal structure on $\mathbb{A}b_{\mathbb{A}^1}(Sm_S)$ given in the Lemma is compatible to the symmetric monoidal structure on $\mathcal{SH}_s^{S^1}(Sm_S)$ induced by the smash-product $(E, F) \mapsto E \wedge F$ [18, 23]; we have the formula for $(M, N) \in (\mathbb{A}b_{\mathbb{A}^1}(Sm_S))^2$

$$M \otimes_{\mathbb{A}^1} N = \pi_0^{\mathbb{A}^1}(HM \wedge HN)$$

6.3. BASE CHANGE AND STABLE $\mathbb{A}^1$-CONNECTIVITY PROPERTY

LEMMA 6.3.1. *The following conditions are equivalent on S:*

(i) The stable $\mathbb{A}^1$-connectivity propertyholds over S;
(ii) For any smooth S-scheme X, the $\mathbb{A}^1$-localization of (X_+) is (-1)-connected.

Proof. Clearly (i) ⇒ (ii). Let's prove the converse implication. Let E be a (-1)-connected S^1-spectrum over S. By Lemma 3.3.4, E is isomorphic in $\mathcal{SH}_s^{S^1}(Sm_S)$ to the telescope of a diagram: $* = E^0 \to \cdots \to E^n \to \dots$ with E^n the cone of a morphism of spectra

$$\vee_\alpha ((X_\alpha)_+)[n_\alpha - 1] \to E^{n-1}$$

where the α's run in some set I_n, with $X_\alpha \in Sm_S$ and $n_\alpha \geqslant 0$.

It suffices to prove that each $L_{\mathbb{A}^1}(E^n)$ is (-1)-connected. We have exact triangles of the form

$$L_{\mathbb{A}^1}(E^{n-1}) \to L_{\mathbb{A}^1}(E^n) \to \vee_\alpha L_{\mathbb{A}^1}((X_\alpha)_+)[n_\alpha]$$

because the $\mathbb{A}^1$-localization functor preserves wedges and exact triangles (and thus telescopes); we conclude easily by induction on n and the assumption. □

LEMMA 6.3.2. (1) *Assume that the stable $\mathbb{A}^1$-connectivity propertyholds over S. Then for any étale morphism $f: S' \to S$ it holds over S'.*

(2) *Assume that $\{U_i \to S\}_i$ is a finite family of étale morphism which is a Nisnevich covering of S. Then the stable $\mathbb{A}^1$-connectivity propertyholds over each of U_i's if and only if it holds over S.*

(3) *Assume $f: S' \to S$ is a morphism which is a filtering limit of a diagram $\{S_\alpha\}_\alpha$ of smooth S-schemes with affine transition morphisms* [14, 8.2] *and that the stable $\mathbb{A}^1$-connectivity propertyholds over each S_α's, then it holds over S'.*

Proof. (1). By the previous Lemma it suffices to prove that the S^1-spectrum $L_{\mathbb{A}^1}(X_+)$ is (-1)-connected for any smooth S'-scheme. It is is clear in view of the assumption and of Lemmas 5.2.4 and 5.2.6 that $L_{\mathbb{A}^1}(f^*(f_\# X)_+)$ is (-1)-connected. But because f is étale, the obvious morphism of smooth schemes over S'

$$X \to f^*(f_\# X) = X \times_S S'$$

admits as retraction the étale morphism $X' := X \times_S S' \to X$ (observe it is not a morphism of S'-schemes). Thus as a smooth S'-scheme X' can be written $X'' \amalg X$. Then $L_{\mathbb{A}^1}(X_+)$ being a summand in $L_{\mathbb{A}^1}(X'_+)$ is (-1)-connected as well.

Now (2) follows from (1) and the fact that an S^1-spectrum E over S is (-1)-connected if and only if each $f_i^*(E)$ is (-1)-connected over U_i.

Again by Lemma 6.3.1, it suffices to prove that the S^1-spectrum $L_{\mathbb{A}^1}(X_+)$ is (-1)-connected for any smooth S'-scheme. By standard results from [14] we know there is an α and a smooth S_α-scheme X_α such that $X \cong X_\alpha \times_{S_\alpha} S'$. We know easily conclude from 5.2.4 and 5.2.6. □

COROLLARY 6.3.3. *The following conditions are equivalent on S:*

(i) The stable $\mathbb{A}^1$-connectivity propertyholds over S;
(ii) The stable $\mathbb{A}^1$-connectivity propertyholds over each local ring of points of S;
(iii) The stable $\mathbb{A}^1$-connectivity propertyholds over each henselian local ring of points of S.

Proof. The implications (i) ⇒ (ii) and (ii) ⇒ (iii) clearly follow from Lemma 6.3.2. The implication (iii) ⇒ (i) follows from the fact that an S^1-spectrum E over S is (-1)-connected if and only if for each $s \in S$, the restriction of E to the local schemes $Spec(\mathcal{O}^h_{S,s}) \to S$ is (-1)-connected, as well as Corollary 5.2.7 (3). □

LEMMA 6.3.4. *Let $f\colon S' \to S$ be a finite morphism. If the stable $\mathbb{A}^1$-connectivity propertyholds over S it holds over S'.*

Proof. By Lemma 5.2.3 and Remark 5.2.5, given a (-1)-connected S^1-spectrum E over S', $f_*(E)$ is (-1)-connected and $L_{\mathbb{A}^1}(f_*(E)) \cong f_*(L_{\mathbb{A}^1}(E))$. The conclusion now follows from the following easy observation: given $x \in X \in Sm_S$, the fiber $f_*(E)(Spec(\mathcal{O}^h_{X,x}))$ is isomorphic to the (finite) wedge

$$\vee_y E(Spec(\mathcal{O}^h_{Y,y}))$$

where $Y = S' \times_S Spec(\mathcal{O}^h_{X,x}))$, a finite $Spec(\mathcal{O}^h_{X,x}))$-scheme which is thus a finite disjoint union of henselian local rings (of smooth S'-schemes) and y runs over the finite set of (closed) points lying over x. □

LEMMA 6.3.5. *Assume that the stable $\mathbb{A}^1$-connectivity propertyholds over S. Then given any S^1-spectrum $E \in Sp^{S^1}(Sm_S)$, the following conditions are equivalent:*

(i) $L_{\mathbb{A}^1}(E)$ is (-1)-connected;
(ii) For any point $x \in S$ the S^1-spectrum $L_{\mathbb{A}^1}(Li_x^* E)$ is (-1)-connected, where $\kappa(x)$ denotes the residue field of x and $i_x\colon Spec(\kappa) \to S$ the canonical morphism.

Proof. The implication (i) ⇒ (ii) follows from Lemma 5.2.4 and Theorem 6.1.8.

To prove (ii) $\Rightarrow$ (i) it suffices to prove that for each point $s \in S$ the restriction

$$L_{\mathbb{A}^1}(E|_{Spec(\mathcal{O}_{S,s})}) \cong \big(L_{\mathbb{A}^1}(E)\big)|_{Spec(\mathcal{O}_{S,s})}$$

of $L_{\mathbb{A}^1}(E)$ to the local scheme $Spec(\mathcal{O}_{S,s})$ is (-1)-connected. We proceed by induction on the Krull dimension d of $Spec(\mathcal{O}_{S,s})$. We let $j: U \to Spec(\mathcal{O}_{S,s})$ denote the complement of the closed point $i: Spec(\kappa(s)) \to Spec(\mathcal{O}_{S,s})$. From Lemma 5.3.2, the following is an exact triangle of S^1-spectra over $Spec(\mathcal{O}_{S,s})$:

$$L_{\mathbb{A}^1}(Lj_{\#}(L_{\mathbb{A}^1}(E|_U))) \to L_{\mathbb{A}^1}(E|_{Spec(\mathcal{O}_{S,s})}) \to i_*(L_{\mathbb{A}^1}(Li_x^* E))$$

We observe that the stable $\mathbb{A}^1$-connectivity propertyholds over the local scheme $Spec(\mathcal{O}_{S,s})$ and over its open subscheme U. The results now follows quite easily because $L_{\mathbb{A}^1}(E|_U)$ is (-1)-connected as we know the restriction of $L_{\mathbb{A}^1}(E)$ to each local rings of U are of smaller dimension. □

We conclude this section by discussing the following Conjecture made by J. Ayoub, which may hopefully give a program to prove more cases of the stable $\mathbb{A}^1$-connectivity property:

CONJECTURE 6.3.6. *(J. Ayoub) For any regular local scheme S, with closed point $i: s \to S$ and open complement $j: U \subset S$, and for any strictly $\mathbb{A}^1$-invariant sheaf M over Sm_U, one has:*

(1) *the sheaf $j_*(M)$ on Sm_S is strictly $\mathbb{A}^1$-invariant;*
(2) *the canonical morphism of sheaves*

$$j_*(M) \to i_*(\pi_0^{\mathbb{A}^1}(Li^*(H(j_*(M)))))$$

is an epimorphism;

(3) *the canonical morphism in $\mathcal{SH}_{\mathbb{A}^1}^{S^1}(Sm_S)$*

$$H(j_*(M)) \to Rj_*(HM) \cong H(Rj_*(M))$$

is an isomorphism.

We observe that $(3) \Rightarrow (1)$ because $Rj_*(HM)$ is $\mathbb{A}^1$-local by construction. We also have the following implications:

LEMMA 6.3.7. *The stable $\mathbb{A}^1$-connectivity propertyfor S implies points (1) and (2) of the conjecture.*

Proof. The sheaf $j_*(M)$ is clearly isomorphic to $\pi_0(Rj_*(HM))$. As $Rj_*(HM)$ is $\mathbb{A}^1$-local by construction, if the stable $\mathbb{A}^1$-connectivity propertyholds over S, $j_*(M)$ is a strictly $\mathbb{A}^1$-invariant sheaf. Moreover, the gluing Lemma 5.3.2, say the spectrum $L_{\mathbb{A}^1}(Lj_\#(HM))$ is the homotopy fiber of the morphism

$$Hj_*(M) \to i_*(L_{\mathbb{A}^1}Li^*(H(j_*(M))))$$

But it is (-1)-connected by the stable $\mathbb{A}^1$-connectivity propertyproperty on S because $Lj_\#(HM)$ is (-1)-connected. This fact then implies (2). □

LEMMA 6.3.8. *The previous conjecture for all local ring of a regular scheme S implies the stable $\mathbb{A}^1$-connectivity propertyfor S.*

Proof. It suffices to prove the stable $\mathbb{A}^1$-connectivity propertyover each local ring of S and proceeding by induction on the dimension of these rings we may assume S itself is local (regular) and that the stable $\mathbb{A}^1$-connectivity propertyholds over the open complement $U \subset S$ of the closed point.

Given a (-1)-connected spectrum E over S, we have the exact triangle

$$L_{\mathbb{A}^1}(Lj_\#(E|U)) \to L_{\mathbb{A}^1}(E) \to i_*(L_{\mathbb{A}^1}Li^*(E))$$

given by the gluing Lemma 5.3.2. The right hand side is (-1)-connected because we know the stable $\mathbb{A}^1$-connectivity propertyover the residue field of S (and because i_* is exact and Li^* preserves (-1)-connected objects). Thus it suffices to check that for a (-1)-connected $\mathbb{A}^1$-local spectrum E over U, $L_{\mathbb{A}^1}(Lj_\#(E))$ is (-1)-connected over S.

Now from the stable $\mathbb{A}^1$-connectivity propertyover U, we see that the homotopy sheaves of E are strictly $\mathbb{A}^1$-invariant sheavesover U. But Rj_* commutes to homotopy inverse limit of towers so that $Rj_*(E)$ is the homotopy inverse limit of the tower $\{Rj_*(E_{\leqslant n})\}_n$. Now assumption (3) implies by induction on n (the number of stages in the Postnikov truncations) that

$$\pi_i(Rj_*(E_{\leqslant n})) = j_*(\pi_i(E_{\leqslant n}))$$

and that moreover this sheaf is strictly $\mathbb{A}^1$-invariant over S. Thus the tower $\{Rj_*(E_{\leqslant n})\}_n$ is exactly the Postnikov tower of its homotopy inverse limit $Rj_*(E)$. This implies also that $i_*(L_{\mathbb{A}^1}Li^*(Rj_*(E)))$ is the homotopy inverse limit of the tower of $i_*(L_{\mathbb{A}^1}Li^*(Rj_*(E_{\leqslant n})))$; this follows from the fact that i_* is exact and Li^* preserves (-1)-connected objects which shows that the homotopy sheaves in this tower stabilize. The gluing Lemma 5.3.2 thus implies that the spectrum $L_{\mathbb{A}^1}(Lj_\#(E))$ is the homotopy inverse limit of the $L_{\mathbb{A}^1}(Lj_\#(E_{\leqslant n}))$ and that moreover the homotopy sheaves in that tower stabilize.

Now the gluing Lemma again and property (2) show that for any n the fiber $L_{\mathbb{A}^1}(Lj_\#(H(\pi_n(E))))$ of

$$H(\pi_n(E)) \to i_*(L_{\mathbb{A}^1} Li^*(H(\pi_n(E))))$$

is (-1)-connected. Using this we get by induction that $L_{\mathbb{A}^1}(Lj_\#(E_{\leqslant n}))$ is (-1)-connected and from the fact that the homotopy sheaves in the tower $\{L_{\mathbb{A}^1}(Lj_\#(E_{\leqslant n}))\}_n$ stabilize we finally get the result. □

6.4. PURE SHEAVES AND THE GERSTEN CONJECTURE

Let us start this last section with the following problem: given a dense open subscheme $U \subset X$ of a smooth S-scheme X, is the S^1-spectrum:

$$L_{\mathbb{A}^1}(X/U)$$

0-connected? If so what is its connectivity? The reason for asking this question comes from the homotopy purity Theorem of [31]. If U is the complement of a closed subscheme $i: Z \to X$ of codimension $d > 0$, smooth over S, then

$$L_{\mathbb{A}^1}(X/U) \cong L_{\mathbb{A}^1}(Th(\nu(i)))$$

where $Th(\nu(i))$ is the Thom space of the normal bundle $\nu(i)$ of the regular immersion i. This can be shown to be $(d-1)$-connected when the stable $\mathbb{A}^1$-connectivity propertyholds over S.

Here is the natural generalization of this fact:

THEOREM 6.4.1. *Assume stable $\mathbb{A}^1$-connectivity propertyholds over S. Let X be a smooth S-scheme and $U \subset X$ an open subscheme such that the complementary closed immersion $Z \to X$ is everywhere of codimension $\geqslant d$ and such that $Z \to S$ is a universally equidimensional morphism (see [38] for instance). Let $X/(X-Z)$ denote the obvious quotient pointed sheaf of sets in the Nisnevich topology on Sm_S and let $(X/(X-Z))$ denote its suspension S^1-spectrum. Then its $\mathbb{A}^1$-localization*

$$L_{\mathbb{A}^1}(X/(X-Z))$$

is a $(d-1)$-connected sheaf of S^1-spectra on Sm_S. In other words the morphism of (strictly $\mathbb{A}^1$-invariant) abelian sheaves

$$\pi_n^{\mathbb{A}^1}((X-Z)_+) \to \pi_n^{\mathbb{A}^1}(X_+)$$

is an isomorphism for $n \leqslant d-2$ and an epimorphism for $n = d-1$.

Remark 6.4.2. For $d=0$, this is exactly Theorem 6.1.8, because the quotient $X/\emptyset$ is indeed the pointed sheaf X_+. The case $d=1$ exactly means that U is dense in X and the statement is that (X/U) is 0-connected as an S^1-spectrum; compare with Lemma 6.1.4.

Proof of the Theorem. By Lemma 6.3.5 and the assumption that Z is universally equidimensional we clearly reduce to proving the theorem when $S=Spec(k)$ is the spectrum of the residue field of a point in S. The conclusion follows now from the equivalences in Lemma 6.4.3 and from Lemma 6.4.4 below. □

The following is rather easy to prove:

LEMMA 6.4.3. *Let $f: E\to F$ be a morphism S^1-spectrum on Sm_S, and denote by $C(f)$ its cone. The following conditions are equivalent*

(1) *The S^1-spectrum $L_{\mathbb{A}^1}(C(f))$ is $(d-1)$-connected.*
(2) *The morphism of (strictly $\mathbb{A}^1$-invariant) abelian sheaves*

$$\pi_n^{\mathbb{A}^1}(E)\to\pi_n^{\mathbb{A}^1}(F)$$

is an isomorphism for $n\leqslant d-2$ and an epimorphism for $n=d-1$.
(3) *For any strictly $\mathbb{A}^1$-invariant sheaf M over Sm_S the homomorphism:*

$$[F; HM[n]]\to[E, HM[n]]$$

is an isomorphism for $n\leqslant d-2$ and a monomorphism for $n=d-1$.

The next Lemma is the crucial point in the proof of Theorem 6.4.1.

LEMMA 6.4.4. *Let $U\subset X$ be an open subscheme of a smooth k-scheme X such that the codimension of the closed complement $X-U$ in X is at least d. Then for any strictly $\mathbb{A}^1$-invariant sheaf M on Sm_k the morphism*

$$H^n_{Nis}(X; M)\to H^n_{Nis}(U; M)$$

is an isomorphism for $n\leqslant d-2$ and a monomorphism for $n=d-1$.

Proof. Assume that k is infinite. Then the cousin complex for X (see [9, (1.3) p. 36]), $U\mapsto E_1^{*,q}(U; M)$ defines a flasque resolution of the Zariski sheaf $\mathcal{H}^q_{Zar}$ associated to $U\mapsto H^p_{Nis}(U; M)$. Indeed if E denotes a fibrant resolution of $H(M)$, the functor $(Sm_k)^{op}\to\mathbb{A}b^*, X\mapsto H^*_{Nis}(X; M)$ defines a cohomology theory with substratum $X\mapsto E(X)\in Sp$ in the sense of [9]; then one gets the result by Corollary 5.1.11 of *loc. cit.*. In particular on has

$$H^n_{Zar}(X,\mathcal{H}^q_{Zar})=H^n(E_1^{*,q}(X; M))$$

where the right hand side is the n-th cohomology group of the Cousin complex.

By Theorem 8.3.1 of *loc. cit.* the left hand side coincides with $H^n_{Nis}(X; M)$ and

$$H^n_{Nis}(X; M) = H^n_{Nis}(X; \mathcal{H}^0_{Nis}) = H^n(E_1^{*,0}(X; M))$$

Now the lemma follows from the long exact sequence in cohomology associated to the epimorphism of (Cousin) complexes $E_1^{p+*,*}(X; M) \to E_1^{p+*,*}(U; M)$ whose kernel vanishes in dimension $\leqslant d-1$ by the assumption on $X-U$.

The case when k is finite, in fact perfect, can be easily deduced from the purity theorem of [31]. See Remark 6.4.5 below. □

Remark 6.4.5. As in Lemma 6.1.3 (see also [27]), when k is perfect field, we can give a quite geometric proof of the previous Lemma. If Z denotes the reduced closed subscheme $X-U$ there is an increasing sequence of reduced closed subschemes:

$$\emptyset \subset F_{dim X} \subset \ldots F_{d+1} \ldots F_d = Z$$

such that each k-scheme $F_s - F_{s+1}$ is smooth and F_s has codimension s in X. By the homotopy purity Theorem of [31], the sheaf $(X - F_{s+1})/(X - F_s)$ is $\mathbb{A}^1$-weakly equivalent to the Thom space of a rank $s \geqslant d$ vector bundle on F_s. But it is easy to check that its Nisnevich cohomology with coefficients in a strictly $\mathbb{A}^1$-invariant sheaf M will vanish in degree $\leqslant d-1$.

The following result is a direct consequence of Theorem 6.4.1. We don't know any "direct" proof of it, in the spirit of the previous proof for fields.

COROLLARY 6.4.6. *Assume stable $\mathbb{A}^1$-connectivity propertyholds over S. Let X be a smooth S-scheme and $U \subset X$ an open subscheme such that the complementary closed immersion $Z \to X$ is everywhere of codimension $\geqslant d$ and such that $Z \to S$ is a universally equidimensional morphism. Then for any strictly $\mathbb{A}^1$-invariant sheaf M on Sm_S the morphism*

$$H^n_{Nis}(X; M) \to H^n_{Nis}(U; M)$$

is an isomorphism for $n \leqslant d-2$ and a monomorphism for $n = d-1$.

Now we address the problem of comparing Zariski and Nisnevich cohomology. We first observe the following result:

LEMMA 6.4.7. *Assume $S = Spec(k)$. Then for any strictly $\mathbb{A}^1$-invariant sheaf M on Sm_k and any $X \in \mathbf{Sm}_k$ the homomorphism*

$$H^*_{Zar}(X; M) \to H^*_{Nis}(X; M)$$

is an isomorphism.

Proof. For k infinite this is known by Theorem 8.3.1 of [9]. Over a finite field k this follows from Lemma 6.4.8 below by reducing as in *loc. cit.* or [35, Section 6] to the case of the infinite field $k(X)$ of rational fractions. □

Observe that in our argument, contrary to [9], we won't use any transfer, even over a finite field.

LEMMA 6.4.8. *Assume $S = Spec(k)$ is the spectrum of a perfect field k. Then for any strictly $\mathbb{A}^1$-invariant sheaf M on Sm_k and any $U \in \mathbf{Sm}_k$ the homomorphisms*

$$H^*_{Nis}(U; M) \to H^*_{Nis}(U_{k(X)}; M|_{k(X)})$$

are monomorphisms.

Proof. By our general base change argument 5.2.7 we see that

$$H^*_{Nis}(U_{k(X)}; M|_{k(X)})$$

is the filtering colimit of the $H^*_{Nis}(U \times (\mathbb{A}^1 - F); M)$, over the ordered set of finite sets F of closed points in $\mathbb{A}^1$. Thus it is sufficient to prove that for any open subscheme $\Omega \subset \mathbb{A}^1$ any closed point $x \in \Omega$ the morphism $H^*_{Nis}(U \times \Omega; M) \to H^*_{Nis}(U \times (\Omega - \{x\}); M)$ is injective (because of course $H^*(U \times \mathbb{A}^1; M) = H^*(U; M)$). For this, it is sufficient to show that the obvious morphism in $\mathcal{SH}^{S^1}_{\mathbb{A}^1}(Sm_S)$: $((\Omega - \{x\})_+) \to (\Omega_+)$ admits a right inverse (so that (Ω_+) is a direct summand of $((\Omega - \{x\})_+)$). By the Mayer–Vietoris triangle

$$((\Omega - \{x\})_+) \to (\Omega_+) \to (\mathbb{A}^1/(\mathbb{A}^1 - \{x\})) \to (\Omega - \{x\}_+)[1]$$

it suffices to show that $(\mathbb{A}^1/(\mathbb{A}^1 - \{x\}) \to (\Omega - \{x\}_+)[1]$ admits a left inverse. But as the obvious composition $(\mathbb{A}^1/(\mathbb{A}^1 - \{x\}) \to ((\Omega - \{x\})_+)[1] \to ((\mathbb{A}^1 - \{x\})_+)[1]$ is easily checked to be the obvious morphism, we reduce to proving that $(\mathbb{A}^1/(\mathbb{A}^1 - \{x\})) \to ((\mathbb{A}^1 - \{x\})_+)[1]$ has a left inverse in $\mathcal{SH}^{S^1}_{\mathbb{A}^1}(Sm_S)$. But there is always a rational point $y \neq x$ in $\mathbb{A}^1$ (in a field $0 \neq 1$), and this easily constructs our splitting. □

Over a general base, the question of whether or not for a strictly $\mathbb{A}^1$-invariant sheaf M on Sm_S and for any $X \in Sm_S$ the homomorphism

$$H^*_{Zar}(X; M) \to H^*_{Nis}(X; M)$$

is an isomorphism is an open question. Now in the next definition, we combine the two properties to get what we called pure sheaves:

DEFINITION 6.4.9. We will say that a sheaf of abelian groups on Sm_S in the Nisnevich topology M is pure if:

(1) for any $X \in Sm_S$ the morphism $H^*_{Zar}(X; M) \to H^*_{Nis}(X; M)$ is an isomorphism;
(2) for any $X \in Sm_S$, any open subscheme $U \subset X$ such that the complementary closed immersion $Z \to X$ is everywhere of codimension $\geqslant d$ the morphism

$$H^n_{Nis}(X; M) \to H^n_{Nis}(U; M)$$

is an isomorphism for $n \leqslant d-2$ and a monomorphism for $n = d-1$.

We have learned that Nisnevich had also considered that property.

EXAMPLE 6.4.10. For instance, if S is normal, any semi-abelian S-scheme $A \to S$ defines a strictly $\mathbb{A}^1$-invariant sheaf which is pure. This follows from the standard properties of abelian schemes [12, Lemma 1] which imply they are flasque sheaves and $\mathbb{A}^1$-invariant.

Lemmas 6.4.4 and 6.4.7 clearly prove:

LEMMA 6.4.11. *Assume $S = Spec(k)$ is the spectrum of a field k. Then any strictly $\mathbb{A}^1$-invariant sheaf M on Sm_k is pure.*

Observe that over a general base S, it is not true that any strictly $\mathbb{A}^1$-invariant sheaf on Sm_S is pure. Take for $i: Z \subset S$ a non-empty closed subscheme of codimension $d > 0$. Then the sheaf $i_*\mathbb{Z}$ is a strictly $\mathbb{A}^1$-invariant (flasque) sheaf on Sm_S, but it is not pure. It is not clear whether or not a pure sheaf of abelian groups is automatically strictly $\mathbb{A}^1$-invariant. Also, it is not clear a priori whether the category of pure strictly $\mathbb{A}^1$-invariant sheavesis abelian or not.

Denote by $\mathcal{SH}^{S^1}_{\mathbb{A}^1}(Sm_S)_{\text{pure}} \subset \mathcal{SH}^{S^1}_{\mathbb{A}^1}(Sm_S)$ the full thick[14] triangulated subcategory generated by suspension S^1-spectra (X_+) of smooth *projective S-schemes*. An object in $\mathcal{SH}^{S^1}_{\mathbb{A}^1}(Sm_S)_{\text{pure}}$ will be called *pure*. We make the following:

CONJECTURE 6.4.12. *The $\mathbb{A}^1$-homotopy sheaves of any pure S^1-spectrum are pure.*

[14]Closed under arbitrary wedges, retracts, and of course cones and suspensions.

Remark 6.4.13. If the conjecture is true, observe that the $\mathbb{A}^1$-homology sheaves of a pure spectrum E are pure as well. To prove this roughly, observe that the $\mathbb{A}^1$-homology sheaves of E are the $\mathbb{A}^1$-homotopy sheaves of the smash product $H\mathbb{Z}\wedge E$, where $H\mathbb{Z}$ the Eilenberg–MacLane spectrum associated to the constant sheaf $\mathbb{Z}$. Now given any S^1-spectrum F over $\mathcal{S}$, and any pure E, a "cellular" decomposition of F will prove that $F\wedge E$ is still pure.

Assuming now S is regular examples of pure sheaves should be the sheaves associated to the presheaves $X\mapsto H^*_{et}(X;M)$ of étale cohomology with coefficients in a locally constant constructible torsion sheaf M on S of torsion prime to each characteristic of the residue fields of S. But this is not yet known unless S itself is smooth over some base field for instance.

Still assuming S is regular, the representability of algebraic K-theory by the Grassmanian [31] would imply that the associated sheaves $\mathcal{K}_n$ to the presheaves of Quillen's K-groups (see Corollary 6) are pure. Also some conjecture of the author predicts that KW can be constructed from smooth projective S-schemes (the orthogonal Grassmanian) so that it would follow that the sheaf $\mathbf{W}$ of Witt groups (see Corollary 6) is pure as well, at least when 2 is invertible in S.

Finally the $\mathbb{A}^1$-homotopy sheaves of the algebraic spheres $(\mathbb{G}_m)^{\wedge n}$, $\mathbb{A}^n-\{0\}$ or $(\mathbb{P}^1)^{\wedge n}$, should be pure over a general regular base scheme. Thus there should exist some type of unramified Milnor K-theory (or Witt groups, etc...) over a general regular base scheme S (with 2 invertible), as these can be obtained over a field by $\mathbb{A}^1$-homotopy sheaves of explicit cones of morphisms between algebraic spheres [29]. In general it seems that most of the interesting cohomology theories are represented by pure spectra.

Pure sheaves and the Gersten conjecture. Let M be a sheaf of abelian groups on Sm_S in the Nisnevich topology. Recall [8,9] that the coniveau spectral sequence for $X\in Sm_S$, for the Nisnevich cohomology of X is a cohomological spectral sequence of the form: $E_r^{p,q}(X;M)\Rightarrow H^{p+q}_{Nis}(X;M)$, and that moreover one can identify the E_1-term with

$$E_1^{p,q}(X;M)=\oplus_{x\in X^{(p)}} H_x^{p+q}(X;M)$$

where $X^{(p)}$ is the set of points in X of codimension p and $H^n_x(X;M)=colim_U\, H^n_{Nis}(U/(U-\overline{x}\cap U);M)$ where U runs over the open subsets which contain x.

We observe that the following Lemma appears in [32]:

LEMMA 6.4.14. *Let $x\in X\in Sm_S$ be any point, n be an integer and M be a sheaf on Sm_S in the Nisnevich topology.*

(1) *Then for* $n \geqslant 2$ *there exists an isomorphism*

$$H^n_x(X; M) \cong H^{n-1}(Spec(\mathcal{O}^h_{X,x}) - \{x\}; M)$$

and an exact sequence

$$0 \to H^0_x(X; M) \to M(Spec(\mathcal{O}^h_{X,x})) \to M(Spec(\mathcal{O}^h_{X,x}) - \{x\}) \to H^1_x(X; M) \to 0$$

Thus in particular $H^n_x(X; M) = 0$ *if* $n > codim(x)$.

(2) *If* M *satisfies condition* (1) *of Definition* 6.4.9 *then moreover, for* $n \geqslant 2$ *there exists an isomorphism*

$$H^n_x(X; M) \cong H^{n-1}(Spec(\mathcal{O}_{X,x}) - \{x\}; M)$$

and an exact sequence

$$0 \to H^0_x(X; M) \to M(Spec(\mathcal{O}_{X,x})) \to M(Spec(\mathcal{O}_{X,x}) - \{x\}) \to H^1_x(X; M) \to 0$$

Proof. (1) The isomorphism and the exact sequence are derived from a base change argument, and the long exact sequence

$$\cdots \to H^{*-1}(Spec(\mathcal{O}^h_{X,x}) - \{x\}; M) \to H^*(Spec(\mathcal{O}^h_{X,x}), Spec(\mathcal{O}^h_{X,x}) - \{x\}; M) \to H^*(Spec(\mathcal{O}^h_{X,x}); M) \to \dots$$

as well as the fact the cohomology of $Spec(\mathcal{O}^h_{X,x})$ is trivial. Moreover as $dim(Spec(\mathcal{O}^h_{X,x}) - \{x\}) = codim(x) - 1$, the second assertion follows from the fact that Nisnevich cohomological dimension is less or equal to the Krull dimension.

Part (2) is derived in the same way in the Zariski topology. □

Thus for any M, the term $E^{p,q}_1(X; M)$ vanishes for $q > 0$.

Assume now that M satisfies condition (2) of Definition 6.4.9. The term $E^{p,q}_1(X; M)$ then, clearly, also vanishes for $q < 0$; in that case, the coniveau spectral sequence is concentrated on the line $q = 0$ and produces an isomorphism between the cohomology of the line $E^{*,0}_1(X; M)$, called the "Cousin complex" [9], and the Nisnevich cohomology $H^*_{Nis}(X; M)$. If moreover, M satisfies condition (1) of Definition 6.4.9 this implies that for the localization at a point of some $X \in Sm_S$ this cousin complex is exact.

It is not difficult to summarize these ideas as follows:

COROLLARY 6.4.15. *Let* M *be a sheaf on* $(Sm_S)_{Nis}$. *Then the following conditions are equivalent:*

(i) *M is a pure sheaf of abelian groups;*
(ii) *For any $x \in X \in Sm_S$, $H^n(\mathcal{O}_{X,x}; M) = 0$ if $n > 0$ and $H^n_x(X; M) = 0$ if $n < codim(x)$;*
(iii) *The Gersten conjecture holds for M and for any localization of a point in Sm_S.*

Remark 6.4.16. Using Lemma 6.4.14 Condition (ii) can be checked to be equivalent to the following:

For any $x \in X \in Sm_S$, $H^n(\mathcal{O}_{X,x}; M) = 0$ if $n > 0$, and

$$H^n(Spec(\mathcal{O}_{X,x}) - \{x\}; M) = 0$$

if $1 \leqslant n < codim(x) - 1$ and

$$M(Spec(\mathcal{O}_{X,x})) \to M(Spec(\mathcal{O}_{X,x}) - \{x\})$$

is injective if $codim(x) > 0$ and surjective if $codim(x) > 1$.

This corollary and the representability of algebraic K-theory by the infinite Grassmanian over a regular base [31] shows that Conjectures 2 and 6.4.12 imply the Gersten conjecture in algebraic K-theory and should also imply in much the same way the Gersten conjecture for sheaves of Witt groups over a regular base in which 2 is invertible.

Brown–Gersten spectral sequences. Now let E be a presheaf of S^1-spectra on Sm_S which satisfies the B.G.-condition and the homotopy invariance. Then for any $X \in Sm_S$ one has also the Brown–Gersten spectral sequence [7] with E_2 term $H^p_{Nis}(X; \mathcal{H}^q(E))$ and converging to $E^{p+q}(X)$; one can check it is exactly the one obtained by the $\mathbb{A}^1$-Postnikov tower of E. In general this spectral sequence won't agree from E_2 with the coniveau spectral sequence for $E^*(X)$. Even for instance in the case $E = HM$ the Eilenberg–MacLane spectrum of an arbitrary strictly $\mathbb{A}^1$-invariant sheaf! However, it is not very hard to check:

LEMMA 6.4.17. *Assume all the stable $\mathbb{A}^1$-homotopy sheaves of the sheaf of spectra associated to E are pure. Then the coniveau spectral sequence agrees from E_2 with the spectral sequence given by the Postnikov tower.*

When $E = K$ represents Algebraic K-theory over a field, this is the well-known Gersten-Quillen spectral sequence [33]. In the case $E = KW$ represent Balmer's Witt groups over a field (of char $\neq 2$) as in [16], this spectral sequence can be shown, using [3], to coincide with the one constructed in [4].

Appendix A. Review of Quillen's Homotopical Algebra

A.1. THE AXIOMS

DEFINITION A.1.1. ([34]) Let $\mathcal{C}$ be a category and $i: X \to Y$ and $p: E \to B$ be morphisms in $\mathcal{C}$. We say that i has the left lifting property (LLP for short) with respect to p or, equivalently, that p has the right lifting property (RLP for short) with respect to i if for any commutative square of the form

$$\begin{array}{ccc} X & \stackrel{f}{\to} & E \\ \downarrow i & & p \downarrow \\ Y & \stackrel{g}{\to} & B \end{array}$$

there exists a morphism $h: Y \to E$ which keeps the diagram commutative, i.e. such that $p \circ h = g$ and $h \circ i = f$ respectively.

DEFINITION A.1.2 ([34]). Let $\mathcal{C}$ be a category equipped with three classes of morphisms (W, C, F) respectively called the *weak equivalences*, the *cofibrations* and the *fibrations*. We say that $(\mathcal{C}, W, C, F)$ is a *model category* (or that (W, C, F) is a model category structure on $(\mathcal{C})$ if the following axioms hold:

- **MC1** $\mathcal{C}$ has all small limits and colimits
- **MC2** If f and g are two composable morphisms and two of f, g or $g \circ f$ are weak equivalences, then so is the third
- **MC3** If the morphism f is retract of g and g is a weak-equivalence, cofibration or fibration then so is f
- **MC4** Any fibration has the right lifting property with respect to *trivial cofibrations*[15] and any *trivial fibration*[16] has the right lifting property with respect to cofibrations
- **MC5** Any morphism f can be functorialy (in f) factorized as a composition $p \circ i$ where p is a fibration and i a trivial cofibration and as a composition $q \circ j$ where q is a trivial fibration and j a cofibration.

Remark A.1.3. Indeed, the previous definition slightly differs from the original one of Quillen: in **MC1** Quillen only assumes the existence of all finite limits and colimits and in **MC5** Quillen only assumes the existence of such factorizations but not the functoriality. It is now recognized that the above axioms makes life easier.

[15] i.e. cofibrations which are also weak equivalences.
[16] i.e. a fibration which is also a weak equivalence.

The associated *homotopy category* is the category $\mathcal{C}[W^{-1}]$ obtained by formally inverting W in $\mathcal{C}$. The previous axioms indeed implies [34] that this category is well defined.

A.2. SIMPLICIAL STRUCTURE

Let $\mathcal{C}$ be a category. We refer the reader to Quillen [34] for the notion of a simplicial structure on $\mathcal{C}$. Recall at least that this means that we are given a functor

$$(\mathcal{C})^{op} \times \mathcal{C} \to \mathcal{C}, (X, Y) \mapsto S(X, Y)$$

with identifications $S_0(X, Y) = Hom_{\mathcal{C}}(X, Y)$ together with a natural transformation

$$S(X, Y) \times S(Y, Z) \to S(X, Z)$$

compatible with the composition in $\mathcal{C}$ and satisfying some axioms.

EXAMPLE A.2.1. If $\mathcal{X}$ and $\mathcal{Y}$ are simplicial sheaves, denote by $S(\mathcal{X}, \mathcal{Y})$ the simplicial set

$$n \mapsto Hom_{\Delta^{op} Shv(Sm_S)}(\mathcal{X} \times \Delta^n, \mathcal{Y})$$

This can be shown to induce a simplicial structure on $\Delta^{op} Shv(Sm_S)$. In much the same way, for sheaves of S^1-spectra E and F denote by $S(E, F)$ the simplicial set

$$n \mapsto Hom_{Sp^{S^1}(Sm_S)}(E \wedge (\Delta^n_+), F)$$

Then this induces a simplicial structure on $Sp^{S^1}(Sm_S)$. These simplicial structures will be refereed to as the standard ones.

With such a simplicial structure fixed, we say that two morphisms $f: X \to Y$ and $g: X \to Y$ are *simplicially homotopic* (with respect to the given simplicial structure) if there is an $h \in S_1(X, Y)$ such that $d_0(H) = g$ and $d_1(H) = f$. We denote by

$$\pi(X, Y)$$

the quotient of the set of morphism $Hom_{\mathcal{C}}(X, Y)$ by the equivalence relation generated by the above simplicial homotopy relation. This set is of course identical to the set $\pi_0(S(X, Y))$ of connected components of the simplicial set $S(X, Y)$.

DEFINITION A.2.2. A *simplicial model category* is a model category $(\mathcal{C}, W, C, F)$ together with a simplicial structure which is *compatible* to the model category structure in the sense that the following axiom holds:

- **SM7** for any cofibration $i: X \to Y$ and any fibration $E \to B$, the obvious map of simplicial sets

  $$S(Y, E) \to S(X, E) \times_{S(X,B)} S(Y, B)$$

  is a Kan fibration [24] which is moreover trivial if either i or p is.

A.3. QUILLEN'S PRINCIPLE OF THE HOMOTOPICAL ALGEBRA

DEFINITION A.3.1. In a model category $(\mathcal{C}, W, C, F)$ an object X is called cofibrant if the canonical morphism $\emptyset \to X$ from the initial object to X is a cofibration, and an object Y is called *fibrant* if the canonical morphism $Y \to *$ from Y to the final object is a fibration.

THEOREM A.3.2. ([34]) *Given a simplicial model category* $(\mathcal{C}, \Delta^\bullet, W, C, F)$ *with associated homotopy category* $\mathcal{H}$ *then for pair* (X, Y) *of a cofibrant object* X *and a fibrant object* Y *the natural map* $Hom_{\mathcal{C}}(X, Y) \to Hom_{\mathcal{H}}(X, Y)$ *induces a bijection*

$$\pi(X, Y) \cong Hom_{\mathcal{H}}(X, Y)$$

Thus we can compute morphisms in the homotopy category of a simplicial model category as follows. Choose a (functorial) trivial fibration $X_c \to X$ with X_c cofibrant (such an X_c is called a *cofibrant resolution* of X). This is possible by the factorization axiom **MC5**. Then in the same way choose a (functorial) trivial cofibration $Y \to Y_f$ with Y_f fibrant (such a Y_f is called a fibrant resolution of Y) and then observe that we have the following sequences of bijections

$$Hom_{\mathcal{H}}(X, Y) \cong Hom_{\mathcal{H}}(X_c, Y) \cong Hom_{\mathcal{H}}(X_c, Y_f) \cong \pi(X_c, Y_f)$$

The first two bijections are completely formal, the last one is a particular case of the previous theorem.

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